

Flows involving Lifshitz solutions

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Abstract

We construct gravity solutions describing renormalization group flows relating relativistic and non-relativistic conformal theories. We work both in a simple phenomenological theory with a massive vector field, and in an $\mathcal{N} = 4$, $d = 6$ gauged supergravity theory, which can be consistently embedded in string theory. These flows offer some further insight into holography for Lifshitz geometries: in particular, they enable us to give a description of the field theory dual to the Lifshitz solutions in the latter theory. We also note that some of the AdS and Lifshitz solutions in the $\mathcal{N} = 4$, $d = 6$ gauged supergravity theory are dynamically unstable.

1 Introduction

The use of gravitational duals to study strongly-coupled field theories [1, 2] has provided a unique calculational tool which has shed light on a number of important questions concerning such field theories. The domain to which this holographic approach has been applied has recently been substantially enlarged to include applications to field theories of interest to condensed matter physics (see [3, 4] for useful reviews). In particular, models have been developed which exhibit an anisotropic scaling symmetry, $t \rightarrow \lambda^z t$, $x^i \rightarrow \lambda x^i$, with z being referred to as the dynamical exponent. Systems with this scaling symmetry can arise as critical points in condensed matter systems. If the field theory has this scaling symmetry, translation and spatial rotations as symmetries, but no boost symmetry, it is commonly referred to as a Lifshitz field theory.

A holographic duality for these field theories was proposed in [5]. The proposal is that the dual of the field theory vacuum is a bulk metric

$$ds^2 = L^2 \left[-e^{2zr} dt^2 + e^{2r} d\vec{x}^2 + dr^2 \right], \quad (1)$$

where L^2 represents the overall curvature scale, and the spacetime has $d + 1$ dimensions, so there are $d - 1$ spatial dimensions \vec{x} . This metric is referred to as a Lifshitz

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geometry; the anisotropic scaling symmetry is realised as an isometry throughout the bulk geometry, analogous to the conformal symmetry in the original AdS/CFT correspondence [1]. Such a metric can be realised as a solution in a variety of bulk gravitational theories with different matter content. In [5], the bulk theory involved two p -form fields with a Chern-Simons coupling. A simpler theory with a massive vector (which is on-shell equivalent to the previous theory) was introduced in [6]. More recently, (1) was realised as a solution in string theory in a number of different truncations [7, 8, 9, 10, 11] (see [12] for earlier attempts).

Here, we will focus both on the phenomenological massive vector theory of [6], which provides the simplest context for studying this geometry, as well as the embedding of four-dimensional Lifshitz geometries in the six-dimensional $F(4)$ gauged supergravity in [8], which provides a realisation in string theory which allows for all values of z , up to issues of flux quantisation.

In both the simple massive vector model of [6] and in the $F(4)$ gauged supergravity [13], there are multiple anti-de Sitter (AdS) and Lifshitz solutions. If we apply the usual holographic dictionary, the anti-de Sitter solutions would be interpreted as dual to the vacuum state in different conformal field theories, and the Lifshitz metrics as dual to the vacuum state in different non-relativistic Lifshitz theories. It is then naturally interesting to investigate the relations between these different field theories.

In this paper, we study this question by constructing domain-wall like solutions which interpolate between the different AdS and/or Lifshitz geometries. These domain wall solutions can be interpreted as dual to renormalization group flows between the corresponding field theories. In the AdS/CFT context, such geometries were first considered in [14] (supersymmetry-preserving flows were considered in [15]). A number of interesting results were obtained, including a holographic c -theorem. Since the theories we are considering have multiple AdS and Lifshitz solutions, there are a variety of possible flows: solutions may interpolate between AdS or Lifshitz in the UV and AdS or Lifshitz in the IR. The simplest case is a spacetime approaching different AdS solutions at large and small distances, corresponding to flows between ordinary relativistic conformal field theories as in [14, 15]. These can be considered as a warm-up exercise for the more complicated cases involving Lifshitz solutions. They also provide a nice simple example exhibiting a type of IR singularity which was recently pointed out in [16]. We will comment briefly on the appearance of these singularities in the context of the gauged supergravity model, but we leave its further investigation for future study.

The flows can provide insight into holographic renormalization for the Lifshitz field theories in a number of ways. Flows which approach an AdS solution in the UV and approach a Lifshitz solution in the IR provide a relation between the more familiar AdS/CFT correspondence and holography for Lifshitz spacetimes. This can be used to understand elements of the holographic interpretation of asymptotically Lifshitz spacetimes (for example, black holes) by considering instead an asymptotically AdS spacetime which approaches Lifshitz at some intermediate distance scale, as in [17]. We also note that a number of the contexts in which Lifshitz solutions have appeared in the recent literature involve such asymptotically AdS solutions, such as [18, 20], or the Lifshitz-like solutions in [19, 20]. Our solutions provide relatively simple examples

of such interpolations. Note some of that the embeddings in string theory which give $z = 2$ Lifshitz solutions [7, 9, 10] do not also have AdS solutions of the same dimension as the Lifshitz solution, so we cannot study such flows in that context (although they may have flows between AdS_{d+1} and d -dimensional Lifshitz geometries). The more elaborate model studied in [11] does however have such AdS solutions in addition to the $z = 2$ Lifshitz solution, and it would be interesting to study this example as well.

If we can identify the conformal field theory dual to the UV AdS solution, this construction can also allow us to define the field theory dual to the IR Lifshitz solution as the corresponding relevant deformation of the former conformal field theory. To identify the dual field theory in this way we need to work in a top-down model; we do not know the field theory dual of the AdS solution in the simple massive vector model. In the gauged supergravity of [13], by contrast, the dual description of the AdS_4 solutions was explored in [21], making use of the twisted field theory ideas of [22]. We show that this twisted field theory construction can easily be extended to provide the first description of a field theory dual to a Lifshitz geometry. A detailed study of this description is left for future work.

Flows which are Lifshitz in the UV provide simple examples of asymptotically Lifshitz spacetimes, corresponding to deformations of the field theory by a relevant operator. These deformations are one of the simplest extensions of the dictionary beyond the consideration of its vacuum state. Examples of such asymptotically Lifshitz solutions were previously constructed in the original paper [5]. We give a systematic discussion of such solutions in the context of the theories we consider.

We study the interpolating solutions both analytically and numerically, matching a linearised perturbation expansion about the AdS and Lifshitz solutions to numerical solutions. In the course of the perturbative analysis, we have noted that both the AdS and Lifshitz solutions in the six-dimensional supergravity of [13] can have excitations that violate the Breitenlohner-Freedman bound [23]. That is, some of these solutions are unstable. We leave exploration of this instability for future work.

The remainder of the paper is organised to first consider the solutions in the simple massive vector theory of [6], and then consider the solutions in the six-dimensional supergravity of [13] following [8]. In the next section, we review relevant aspects of the massive vector theory, and identify its AdS and Lifshitz solutions. In section 3, we construct domain walls dual to renormalization group flows interpolating between these solutions. We first consider the linearised analysis about each of the AdS and Lifshitz solutions, and then construct the full interpolating solutions numerically. In section 4, we discuss the six-dimensional supergravity theory and the four-dimensional AdS and Lifshitz solutions obtained from compactification on a compact hyperbolic space with flux, and obtain a consistent truncation to a four-dimensional theory. We then discuss interpolating between these solutions in section 5. In this section we also discuss the identification of field theories dual to the Lifshitz solutions. We conclude with some remarks and discussion of open problems in section 6.

2 Massive vector theory

The simplest context in which to consider the Lifshitz metric is the massive vector theory introduced in [6]. This is a phenomenological model, so it is not *a priori* clear that there is a well-defined quantum theory of gravity that reduces to this theory in a low-energy limit.¹ That is, it is not clear that the Lifshitz geometry here is genuinely dual to a well-defined quantum field theory. Nonetheless, this is a simple theory, so it provides a useful warmup before we turn to the more complicated theory considered in [8], and it will turn out that the structure of the interpolating solutions in these two theories is actually strikingly similar.

The bulk spacetime action for the massive vector theory is

$$S = -\frac{1}{16\pi G} \int d^{d+1}x \sqrt{-g} (R - 2\Lambda - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu). \quad (2)$$

To have the solution (1), we need the cosmological constant and mass to be related to the dynamical exponent z by

$$\Lambda = -\frac{1}{2L^2} (z^2 + (d-2)z + (d-1)^2), \quad m^2 L^2 = (d-1)z, \quad (3)$$

which implies

$$\frac{\Lambda}{m^2} = -\frac{1}{2(d-1)z} (z^2 + (d-2)z + (d-1)^2). \quad (4)$$

The theory then has a solution with metric (1) and

$$A = \alpha e^{zr} dt = L \sqrt{\frac{2(z-1)}{z}} e^{zr} dt. \quad (5)$$

Since (4) is quadratic in z , if we regard the Lagrangian parameters Λ , m as fixed, there will be generically either two or no real solutions for z . The quadratic has two real roots for $\Lambda < -\frac{(3d-4)}{2(d-1)} m^2$. If we call the smaller root z_1 , the larger root will be $z_2 = (d-1)^2/z_1$. The form of the vector field (5) restricts us to considering only solutions with $z > 1$, and we find that $z_2 > 1$ for all d , whereas $z_1 > 1$ for $\Lambda/m^2 > -d/2$.

Thus, this theory will have a single Lifshitz solution for $\Lambda/m^2 \leq -d/2$, two Lifshitz solutions for $-d/2 < \Lambda/m^2 < -\frac{(3d-4)}{2(d-1)}$, a single degenerate solution with $z = d-1$ for $\Lambda/m^2 = -\frac{(3d-4)}{2(d-1)}$, and none for $\Lambda/m^2 > \frac{(3d-4)}{2(d-1)}$. It also has an AdS_{d+1} solution with no vector field for all $\Lambda < 0$. These different solutions are depicted in figure 1. The flows depicted in this figure will be explained in the next section.

¹Qualitatively similar models with additional scalar fields can be embedded in string theory, as in [9], but the presence of the additional scalar fields can make significant differences to the physics; for example, the black hole solutions of [24] are quite different from the ones in the massive vector model obtained in [25].

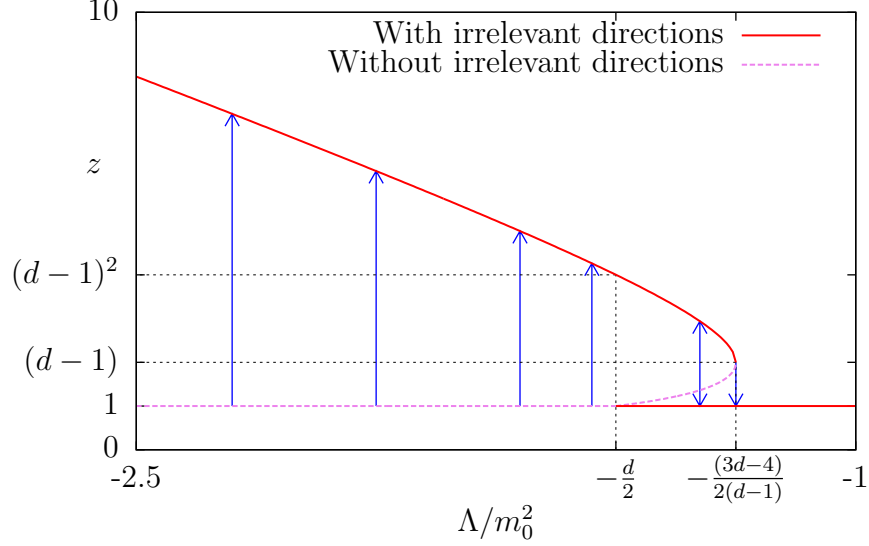


Figure 1: The vacua of the massive vector model in $d+1$ dimensions, labelled according to whether or not they possess an irrelevant perturbation within our ansatz. This plot was made using $d = 3$, but is qualitatively the same at higher d . The arrows denote the holographic RG flows.

3 Flows in the massive vector theory

We now consider the construction of domain wall geometries which interpolate between these different solutions. Such a solution was previously found in [5], who numerically found a spacetime that is asymptotically AdS at small r and asymptotically Lifshitz with $z = 2$ at large r . Since the theory considered in [5] is on-shell equivalent to the massive vector model, their solution will be a special case of the solutions we find here.² Our aim in this section is to extend this to give a comprehensive survey of the flows relating all the different solutions for arbitrary z and d .

Since all of the solutions preserve translation invariance and spatial rotations, we assume that these symmetries are preserved in the domain wall solutions, and hence consider an ansatz

$$ds^2 = L^2 \left(-e^{2F(r)} dt^2 + e^{2D(r)} \frac{dr^2}{r^2} + r^2 \sum_{i=1}^{d-1} dx_i^2 \right), \quad (6)$$

$$A = \alpha(r) e^{F(r)} dt, \quad (7)$$

²Note that the flow considered in [5] is a somewhat special case - we see from figure 1 that this is at the value of Λ/m^2 for which there is only one Lifshitz solution.

where L is given by (3). The equations of motion are:

$$\frac{(d-1)}{L^2 e^{2D}} [rD' - d] = \frac{(\alpha' + \alpha F')^2 r^2}{L^4 e^{2D}} + \frac{m^2 \alpha^2}{4L^2} + \Lambda \quad (8)$$

$$-\frac{(d-1)}{L^2 e^{2D}} [rF' - (d-2)] = \frac{(\alpha' + \alpha F')^2 r^2}{L^4 e^{2D}} - \frac{m^2 \alpha^2}{4L^2} + \Lambda \quad (9)$$

$$m^2 L^2 e^{2D} \alpha = r^2 (\alpha' + \alpha F')' + r (\alpha' + \alpha F') (d - rD') \quad (10)$$

plus a Bianchi identity not shown here. The Lifshitz solutions correspond to $F = z \ln r$, $D = 0$, $\alpha = L \sqrt{\frac{2(z-1)}{z}}$, and the AdS solution is the $z \rightarrow 1$ limit of the Lifshitz solution.

3.1 Linearized equations of motion

We start by considering the linearised equations of motion around each of these solutions. This will enable us to identify the conformal dimensions of the corresponding operators in the field theory. Around the AdS solution, the linearisation is very simple, as many of the equations decouple. If we write $F = \ln r + \delta F$, the linearised equations of motion are

$$r\delta F' = -rD' = dD, \quad r(r\alpha')' = (L^2 m^2 - (d-1))\alpha - d r \alpha', \quad (11)$$

where prime denotes derivatives with respect to r , and the solutions are

$$\delta F = F_0 + F_1 r^{-d}, \quad D = -F_1 r^{-d}, \quad \alpha = \alpha_1 r^{-\Delta_1} + \alpha_2 r^{-\Delta_2}, \quad (12)$$

where

$$\Delta_{1,2} = \frac{d}{2} \mp \frac{\sqrt{4m^2 L^2 + (d-2)^2}}{2}. \quad (13)$$

The constant F_0 corresponds to the timelike component of the boundary metric in the field theory. The mode F_1 corresponds to the expectation value of the field theory energy density (more precisely, the tracelessness of the boundary stress tensor implies a relation between the boundary energy density and pressure, so this mode represents a non-zero expectation value for both energy density and pressure). If we impose boundary conditions on the vector field that fix the slow fall-off mode α_1 , the α_1, α_2 modes correspond respectively to the source and expectation value for the operator dual to the massive vector. The dimension of this operator is

$$\Delta = \frac{d}{2} + \frac{\sqrt{4m^2 L^2 + (d-2)^2}}{2}. \quad (14)$$

For $m^2 L^2 < (d-1)$, the operator dual to the massive vector is relevant. This corresponds to $\Lambda/m^2 < -d/2$. Thus for $\Lambda/m^2 < -d/2$, deforming the conformal field theory dual to the AdS solution by this relevant operator will generate a flow from this theory in the UV. In the bulk, the interpolating solution corresponding to this RG flow will be an asymptotically AdS spacetime with a perturbation with non-zero

α_1 at large r . Since this corresponds to turning on the vector field which sources the Lifshitz solutions, the natural expectation is that this flow will approach Lifshitz in the IR. For this range of parameters, there is a unique Lifshitz solution. This flow is indicated by the vertical arrows to the left in figure 1, and will be constructed numerically below. Note that such flows from AdS in the UV to Lifshitz in the IR have not previously been constructed for this theory.

For the Lifshitz solutions, the analysis of the linearised equations of motion was performed for $d = 3$ in [26, 27]. Here we extend this analysis to general d . The solutions we are interested in here correspond to the scalar parts of the constant perturbations in the previous analysis. We write $F = z \ln r + \delta F$, $\alpha = \alpha_0(1 + \delta\alpha)$, with α_0 the background value given by (5). There is then a simple two-parameter set of solutions given by

$$\delta F = F_0 + \frac{d-1-z}{d-1+z} \frac{F_1}{r^{z+d-1}}, \quad D = \frac{F_1}{r^{z+d-1}}, \quad \delta\alpha = -\frac{d-2+z}{z-1} \frac{F_1}{r^{z+d-1}}. \quad (15)$$

As before, F_0 corresponds to the source for and F_1 corresponds to the expectation value of the field theory energy density, as discussed in detail for $d = 3$ in [27].³ This is a marginal operator, since in a Lifshitz solution, the dimension of a marginal operator is $z + (d-1)$, because of the different scaling of the time direction.

We have two other solutions, given by

$$\delta F = \alpha_1 \frac{(z+d-2)(z+d-1+\beta)}{r^{\frac{1}{2}(z+d-1-\beta)}} + \alpha_2 \frac{(z+d-2)(z+d-1-\beta)}{r^{\frac{1}{2}(z+d-1+\beta)}} \quad (16)$$

$$D = \alpha_1 \frac{(z-1)(z-3d+3+\beta)}{r^{\frac{1}{2}(z+d-1-\beta)}} + \alpha_2 \frac{(z-1)(z-3d+3-\beta)}{r^{\frac{1}{2}(z+d-1+\beta)}} \quad (17)$$

$$\delta\alpha = \alpha_1 \frac{(z+d-2)(3z-d+1-\beta)}{r^{\frac{1}{2}(z+d-1-\beta)}} + \alpha_2 \frac{(z+d-2)(3z-d+1+\beta)}{r^{\frac{1}{2}(z+d-1+\beta)}} \quad (18)$$

where

$$\beta(z, d)^2 = (z+d-1)^2 + 8(z-1)(z-d+1). \quad (19)$$

We would expect to interpret these as the source for and the expectation value of the operator dual to the massive vector field. We will assume this interpretation is valid, but note that in $d = 3$ there were some unanswered questions about the calculation of the expectation value for $z \geq 2$ [27]. This then corresponds to an operator of dimension

$$\Delta = \frac{1}{2}(z+d-1+\beta(z, d)). \quad (20)$$

Thus, this operator is relevant if $\beta(z, d) < z+d-1$, that is for $z < (d-1)$. It is always a relevant perturbation on the branch of Lifshitz solutions with the smaller value of z , and always an irrelevant perturbation on the branch with the larger value.

The perturbation of the smaller z Lifshitz solution at large r by the mode α_1 which corresponds to the source for this operator then sources a flow to the IR. This perturbation represents the leading deformation of the massive vector field, so

³The geometrical description of the boundary data F_0 was recently discussed in [28].

depending on the sign, we expect this to terminate either at the AdS solution or the larger z Lifshitz solution, which has a larger value for the vector. We also see that the larger z solution has an irrelevant direction, which we expect to correspond to the flow from AdS or smaller z . These expectations are confirmed in the next subsection.

3.2 Numerical flows

We now turn to numerics to confirm the existence of the interpolating geometries predicted from our analysis of the linearised equations of motion in the previous section. As in previous work, starting with [15], we construct these solutions by starting from the IR (small r region of the geometry). This is convenient because we want to consider geometries dual to renormalization group flows, so we are not interested in exciting the modes corresponding to the expectation value of the dual operators. Since those modes grow towards small r (the IR), we can most easily construct the flows numerically by starting from a candidate IR geometry at small r and following the effect of a small deformation by an irrelevant operator as we integrate out to larger r . Since the linearised analysis told us that within the ansatz we are considering these solutions have at most a single irrelevant direction, all we can choose is the sign of the perturbation.

The AdS solutions had an irrelevant perturbation for $\Lambda/m^2 > -d/2$. Based on the analysis presented in figure 1, we expect there to be flows from the small z Lifshitz solution at large r which approach the AdS solution at small r along this deformation. The flow found in [5] in the case $z = 2$ $d = 3$ is a special case of this class. Since the AdS vacuum has $\alpha = 0$ and the irrelevant perturbation involves only the vector field, the $\alpha \mapsto -\alpha$ symmetry means that the sign of the perturbation does not matter here.

We have numerically constructed examples of such flows for $d = 3, 4, 5$. An example with $z = 1.6$ in $d = 3$ is shown in figure 2. The flows typically rapidly approach the Lifshitz solution at large r , although in the $z = 2$ case it is much slower. This is to be expected, since in this case the direction we are approaching the Lifshitz point along is marginal at the linear level.

The Lifshitz solution with larger z has an irrelevant perturbation, so we would expect to be able to construct solutions which approach this solution at small r ; from the analysis in figure 2, we expect that for $\Lambda/m^2 > -d/2$ this will be a flow from the small z Lifshitz solution at large r (if we choose the sign of the perturbation at small r to be in the direction of decreasing α). These will therefore be examples of Lifshitz to Lifshitz flows. In figure 3 we show an example of such a flow for $d = 3$. We have also constructed examples in $d = 4, 5$ which are qualitatively similar.

For $\Lambda/m^2 < -d/2$, we expect the irrelevant deformation around the large z Lifshitz solution to be associated with a flow from the AdS solution at large r (again assuming we choose the sign of the perturbation at small r to be in the direction of decreasing α). We numerically constructed examples in $d = 3, 4, 5$; an example with $d = 3$ is shown in figure 4.

Examples of flows from Lifshitz in the UV to AdS in the IR were previously constructed in [5], but the other two types of flow solutions we have constructed here, from Lifshitz to Lifshitz and from AdS to Lifshitz, are new. The latter are probably

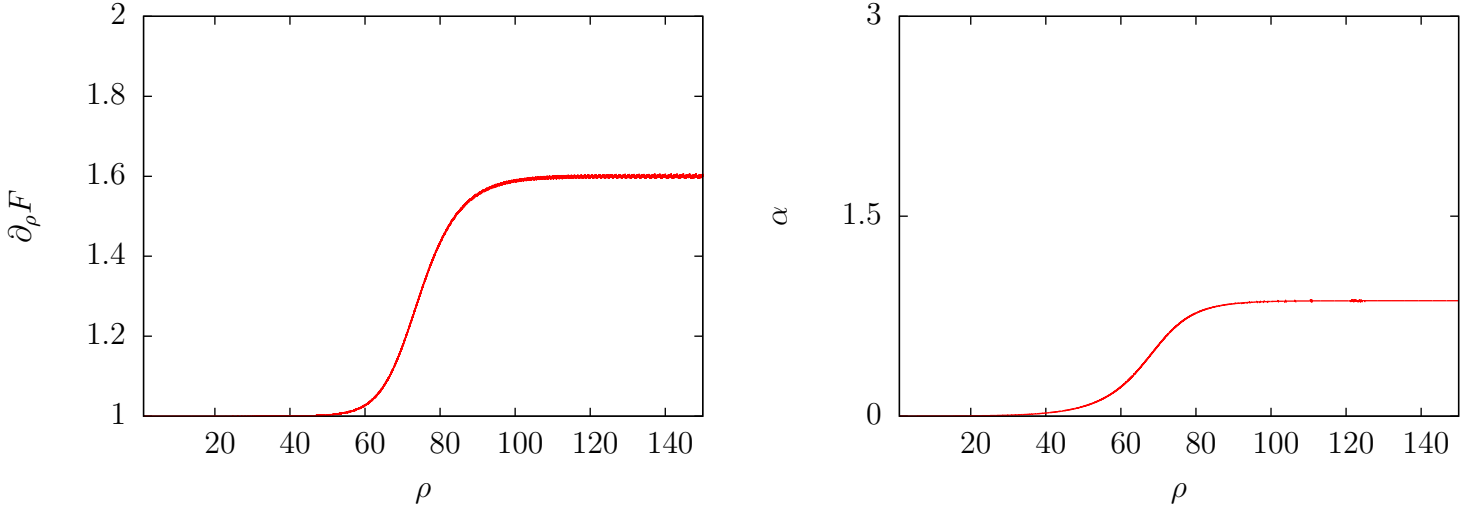


Figure 2: Holographic RG flow in $d = 3$ from a Lifshitz spacetime with $z = 1.6$ in the UV to an AdS_4 spacetime in the IR. In our numerical analysis we use a radial coordinate $\rho = \ln r$. Note that $\partial_\rho F \rightarrow z$ as we approach one of the AdS or Lifshitz solutions.

the most interesting. In the context of this simple massive vector model, these AdS to Lifshitz flows are interesting primarily for the potential to relate the study of the holographic dictionary in Lifshitz geometries to the better understood AdS case, by embedding asymptotically Lifshitz geometries in asymptotically AdS ones, and hence relating the calculation e.g. of correlation functions in the Lifshitz context to observables in the UV conformal field theory. The application to understanding the field theory dual to the Lifshitz geometry is hampered here by our lack of an understanding of the field theory dual to the AdS solutions in this massive vector model. This motivates us to turn in the next section to the consideration of Lifshitz solutions in a supergravity theory which can be embedded into string theory, where we can obtain a concrete interpretation of the geometries in terms of dual field theory.

4 $F(4)$ gauged supergravity

We now turn to the consideration of a more complicated theory, which can be embedded into string theory, the six-dimensional $\mathcal{N} = 4$ $F(4)$ gauged supergravity of [13]. This theory can be obtained as a consistent truncation of a Kaluza-Klein reduction of massive type IIA supergravity [29], so solutions of this theory can be uplifted to solutions of string theory in a background including D8- brane charge.

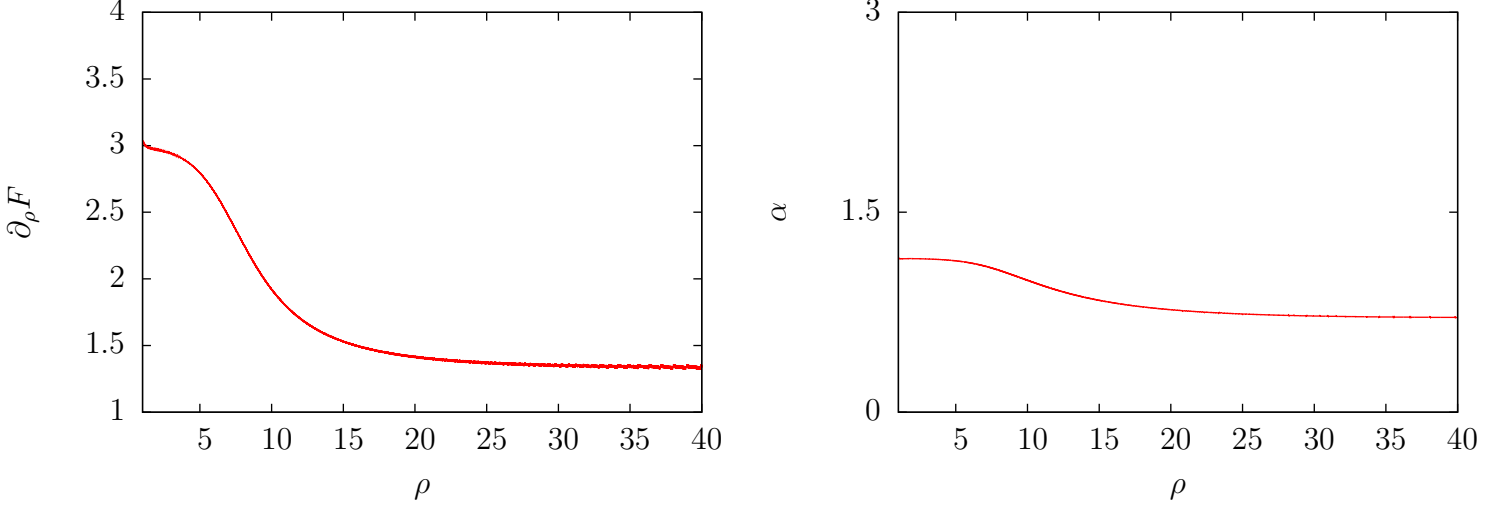


Figure 3: Holographic RG flow in $d = 3$ from a Lifshitz spacetime with $z = 1.333$ in the UV to one with $z = 3$ in the IR. In our numerical analysis we use a radial coordinate $\rho = \ln r$. Note that $\partial_\rho F \rightarrow z$ as we approach one of the AdS or Lifshitz solutions.

The bosonic part of the action for this theory is

$$\begin{aligned}
S = \int d^6x \sqrt{-g} & \left[\frac{1}{4} R - \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{1}{4} e^{-\sqrt{2}\phi} (\mathcal{H}^{\mu\nu} \mathcal{H}_{\mu\nu} + F^{(i)\mu\nu} F_{\mu\nu}^{(i)}) \right. \\
& - \frac{1}{12} e^{2\sqrt{2}\phi} G_{\mu\nu\rho} G^{\mu\nu\rho} + \frac{1}{8} (g^2 e^{\sqrt{2}\phi} + 4gme^{-\sqrt{2}\phi} - m^2 e^{-3\sqrt{2}\phi}) \Big] \\
& + \frac{1}{8} \int d^6x \left(B \wedge \mathcal{F} \wedge \mathcal{F} + mB \wedge B \wedge \mathcal{F} + \frac{m^2}{3} B \wedge B \wedge B + B \wedge F^{(i)} \wedge F^{(i)} \right).
\end{aligned} \tag{21}$$

We follow the conventions of [13]; the bosonic fields are the metric $g_{\mu\nu}$, the dilaton ϕ , the two-form $B_{\mu\nu}$, an $SU(2)$ gauge field $A_\mu^{(i)}$ and a $U(1)$ gauge field \mathcal{A}_μ . The field strengths are

$$\mathcal{F} = d\mathcal{A}, \quad F^{(i)} = dA^{(i)} + g\epsilon^{ijk} A^{(j)} \wedge A^{(k)}, \quad G = dB, \tag{22}$$

and we write

$$\mathcal{H}_{\mu\nu} = \mathcal{F}_{\mu\nu} + mB_{\mu\nu}. \tag{23}$$

The Lagrangian involves two parameters, g and m . We consider only $g > 0$, $m > 0$, referred to as $\mathcal{N} = 4^+$ in the notation of [13]. Note that as explained in [13], there is a freedom to make field redefinitions which relates different theories; the inequivalent theories are labeled by the ratio g/m .

We want to consider the AdS and Lifshitz solutions in this theory. The theory of course also has an AdS_6 solution, discussed in detail in [13]. We will very briefly review this solution, as we will be considering some flows involving asymptotically

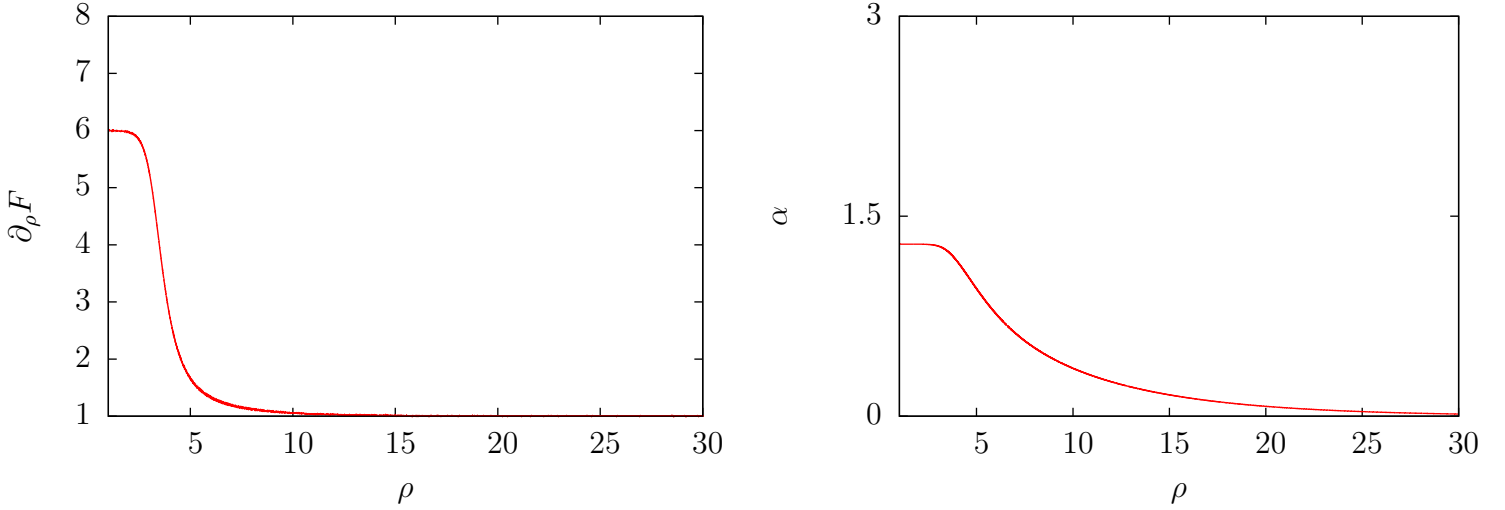


Figure 4: Holographic RG flow in $d = 3$ from an AdS_4 spacetime in the UV to a Lifshitz spacetime with $z = 6$ in the IR. In our numerical analysis we use a radial coordinate $\rho = \ln r$. Note that $\partial_\rho F \rightarrow z$ as we approach one of the AdS or Lifshitz solutions.

AdS_6 solutions, but we focus on describing the four-dimensional AdS and Lifshitz solutions obtained by considering a further compactification of this theory on a compact hyperbolic space.

The AdS_6 solution has metric

$$ds^2 = r^2 \eta_{\alpha\beta} dx^\alpha dx^\beta + L^2 \frac{dr^2}{r^2}, \quad (24)$$

the vector and two-form fields vanishing, and dilaton $\phi = \phi_0$ with $e^{-2\sqrt{2}\phi_0} = \frac{g}{m}$ or $e^{-2\sqrt{2}\phi_0} = \frac{g}{3m}$. The latter case is supersymmetric, and is dual to the conformal field theory obtained in the IR limit of the $\mathcal{N} = 2$ $Sp(N)$ gauge theory on the worldvolume of D4-branes in the presence of D8-branes [30, 31].

The four-dimensional solutions are obtained by considering a compactification of this six-dimensional theory on a compact hyperbolic space. To describe the AdS and Lifshitz solutions, we can take the metric to have the form

$$ds^2 = -e^{2F(r)} dt^2 + r^2 d\vec{x}^2 + e^{2d_0} \frac{dr^2}{r^2} + e^{2h_0} d\Omega_2^2, \quad (25)$$

where d_0, h_0 are constants, and

$$d\Omega_2^2 = \frac{1}{y_2^2} (dy_1^2 + dy_2^2) \quad (26)$$

is the metric on H^2 , a two-dimensional space of constant negative curvature. We take the global geometry of this two-dimensional space to be some compact quotient of H^2 by a discrete subgroup Γ of its isometry group. For an AdS_4 solution, $F(r) = \ln r$,

while for a Lifshitz solution, $F(r) = z \ln r$. We take the dilaton to be a constant, $\phi = \phi_0$, and for the vector and two-form fields, we take

$$F^{(3)} = \alpha_0 \frac{e^{F(r)+d_0}}{r} dt \wedge dr + \frac{\gamma}{y_2^2} dy_1 \wedge dy_2, \quad (27)$$

and

$$\mathcal{H} = mB = m \frac{\bar{\beta}_0}{2} r^2 dx_1 \wedge dx_2, \quad (28)$$

so we consider a flux of one component of the $SU(2)$ gauge field on the compact space.

The equations of motion fix $\alpha_0 = \gamma \bar{\beta}_0 e^{\sqrt{2}\phi_0} e^{-2h_0}$. Charge conservation implies γ is fixed (in particular, in the interpolating solutions it will remain a constant), and from the four-dimensional point of view it corresponds to a parameter labeling the theory rather than a feature of a particular solution. Thus, in determining Lifshitz solutions, we should look to solve for the other parameters z , d_0 , h_0 , ϕ_0 and $\bar{\beta}_0$ in terms of g , m and γ . As was observed in [8], in this ansatz there is a further freedom to rescale fields additional to the field redefinition of [13]; as a result both g and m just set an overall scale for fields. It will be convenient for us to define a slightly different set of rescaled variables to those considered in [8]. We define

$$\varphi_0 = \sqrt{\frac{m}{g}} e^{-\sqrt{2}\phi_0}, \quad e^{-2H_0} = \frac{\gamma}{\sqrt{gm}} e^{-2h_0}, \quad e^{2D_0} = \sqrt{g^3 m} e^{2d_0}, \quad \beta_0 = \sqrt{\frac{m}{g}} \bar{\beta}_0. \quad (29)$$

AdS₄ solutions were discussed in [13] and more recently in [21]. They have $z = 1$ and $\beta_0 = 0$. Solving the equations of motion then gives us a relation between the flux γ and the constant values φ_0 , H_0 and D_0 . Surprisingly, solutions only exist for a certain range of values of $g^2 \gamma^2$. Solving the equations of motion gives

$$g^2 \gamma^2 = \frac{(1 - \varphi_0^2)(3\varphi_0^2 - 1)}{(1 - 2\varphi_0^2 + 2\varphi_0^4)^2}, \quad (30)$$

which has two solutions for $0 < g^2 \gamma^2 < \frac{9 - \sqrt{216}}{\sqrt{1536 - 44}} \approx 1.185$ (one either side of $\varphi_0^2 = 1 - 1/\sqrt{6} \approx 0.592$; note $\varphi_0 > 0$ for real ϕ_0), and no solutions for larger $g^2 \gamma^2$. We will refer to these as the small φ_0 and large φ_0 solutions; the small φ_0 solution has $\varphi_0^2 \in \left(\frac{1}{3}, 1 - \frac{1}{\sqrt{6}}\right)$ and the large φ_0 solution has $\varphi_0^2 \in \left(1 - \frac{1}{\sqrt{6}}, 1\right)$. The values of the other fields at the AdS₄ solution are most conveniently written in terms of φ_0 ,

$$e^{-2D_0} = \frac{\varphi_0(2 - \varphi_0^2)}{6}, \quad e^{-2H_0} = \frac{\sqrt{(1 - \varphi_0^2)(3\varphi_0^2 - 1)}}{2\varphi_0}. \quad (31)$$

Note that these are both positive for all $\varphi_0^2 \in \left(\frac{1}{3}, 1\right)$, see figure 5.

For $\varphi_0^2 = \frac{1}{2}$, these solutions preserve half the supersymmetry of the original six-dimensional theory. In [21], these supersymmetric AdS₄ solutions were related to the conformal field theory obtained by taking the low-energy limit of a five-dimensional

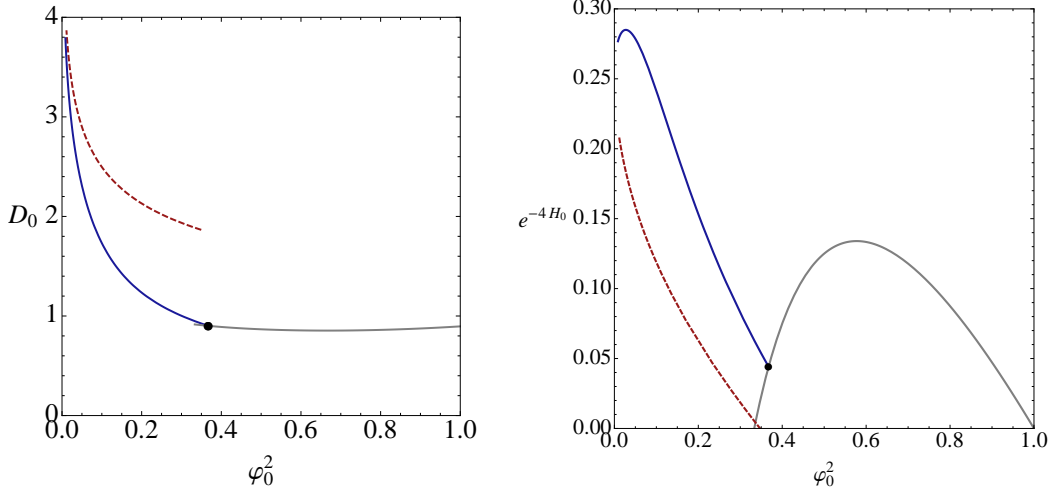


Figure 5: Plots of D_0 and H_0 for the AdS and Lifshitz solutions as a function of φ_0^2 . The AdS solutions are shown in grey, the smaller Lifshitz in blue, and the larger z Lifshitz in dashed red.

twisted field theory. We will describe the flows corresponding to this IR limit in the next section.

In [8], Lifshitz solutions with arbitrary z were obtained in this ansatz. These Lifshitz solutions break all of the supersymmetry of the theory. Solving the equations of motion for $z \neq 1$ gives

$$g^2\gamma^2 = \frac{(z+4)[(z+2)(z-3) \pm 2\sqrt{2z+8}]}{[3z+6 \mp 2\sqrt{2z+8}]^2}, \quad (32)$$

which fixes z for given $g^2\gamma^2$, and the other parameters are given by

$$\varphi_0^2 = \frac{z^2(z+4)}{6+z \mp 2\sqrt{2(z+4)}}, \quad \beta_0 = \varphi_0\sqrt{z-1}, \quad (33)$$

and

$$e^{4D_0} = 4z\sqrt{(4+z)^3[6+z \mp 2\sqrt{2(z+4)}]}, \quad e^{-2H_0} = g\gamma e^{-2D_0}(6+3z \mp 2\sqrt{2(z+4)}). \quad (34)$$

These different solutions are shown in figure 5. There is a single sign choice here; we can choose either the upper or the lower sign in all expressions to obtain a solution. Thus, for given $g^2\gamma^2$, there are two possible values for z , and the other fields are then uniquely specified once one of these two values is chosen. As the lower sign gives larger values of z , we refer to this as the larger z branch of solutions, and the upper sign as the smaller z branch.

We are restricted to solutions with $z \geq 1$ by (33). There are then solutions on the larger z branch for all values of $g^2\gamma^2$; z increases monotonically from $z = 4.29$ at $g^2\gamma^2 = 0$. The solutions on the smaller z branch have $z = 1$ at $g^2\gamma^2 = \frac{5(-6+2\sqrt{10})}{(9+2\sqrt{10})^2} \approx 0.23$ and again have z increasing monotonically with $g^2\gamma^2$.

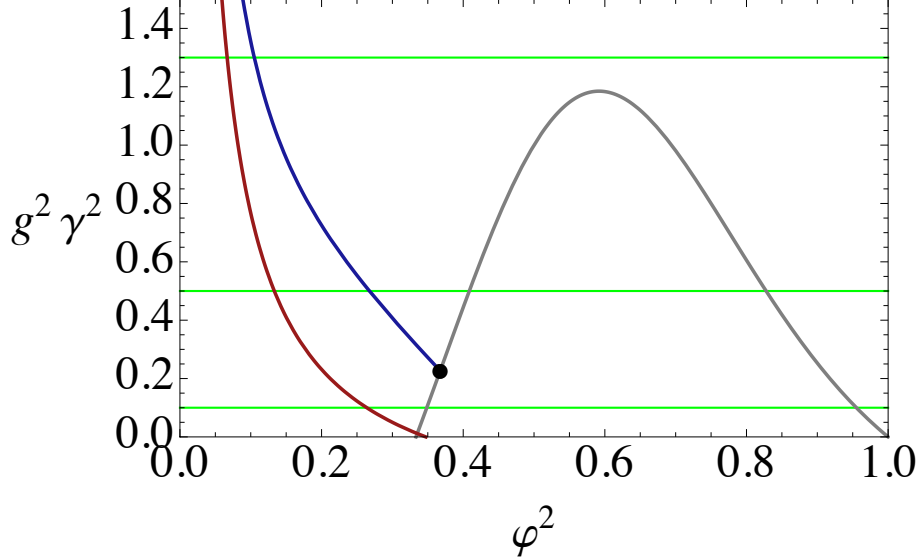


Figure 6: The possible AdS and Lifshitz solutions for the gauged supergravity theory as a function of the flux $g^2\gamma^2$. Reading from left to right, the curves represent the large z Lifshitz solution, the small z Lifshitz solution and the AdS solutions. Flows interpolating between these solutions will be at fixed $g^2\gamma^2$, so the horizontal lines indicate possible flows. As discussed in the text, the structure of the expected flows is similar to that shown in figure 1 for the massive vector case; the main difference is the appearance of two AdS solutions here.

In summary, there are two AdS solutions for $g^2\gamma^2 < 1.18$, and none for larger values. There is one Lifshitz solution for $g^2\gamma^2 < 0.23$, with $z > 4.29$; there are two Lifshitz solutions for $g^2\gamma^2 > 0.23$, with the second solution starting from $z = 1$ at $g^2\gamma^2 = 0.23$, where it coincides with one of the two AdS solutions. This structure is reminiscent of what we saw for the massive vector model earlier, although these Lifshitz solutions never meet. The solutions are plotted in figure 6.

4.1 Consistent truncation

We are interested in studying solutions which interpolate between the fixed point AdS and Lifshitz solutions identified above. It is straightforward to study these solutions in terms of the six-dimensional theory, and retaining this point of view will prove useful for understanding the field theory interpretation of the solutions later. However, examining the solutions from the point of view of a Kaluza-Klein reduction provides a complementary viewpoint which also provides useful insight. Since all the solutions we are considering excite only the overall volume of the internal space, this reduction is fairly simple, and we give here a consistent truncation of the reduced equations of motion which will include the solutions of interest.

We consider a metric ansatz

$$ds_6^2 = e^{-2h} g_{\mu\nu} dx^\mu dx^\nu + e^{2h} d\Omega_2^2, \quad (35)$$

where $d\Omega_2^2$ is given in (26), $g_{\mu\nu}$ is an arbitrary four-dimensional metric, and h is a function of the x^μ . For the matter fields, we take $\mathcal{F} = 0$,

$$F_6^{(3)} = F + \gamma\epsilon_2, \quad B_6 = B + B_0\epsilon_2. \quad (36)$$

The dilaton ϕ is a function of the x^μ . Here ϵ_2 is the volume form on the internal space, and F, B are two-forms on the four-dimensional spacetime. The equations of motion imply γ is a constant, but B_0 is a function of the x^μ .

This ansatz will satisfy the six-dimensional equations of motion following from the action (21) if $g_{\mu\nu}, h, \phi, B_0, B_{\mu\nu}$ and $F_{\mu\nu}$ satisfy the equations of motion of the action

$$\begin{aligned} S = \int d^4x \sqrt{-g} & \left[\frac{1}{4}R - \frac{1}{12}e^{2\sqrt{2}\phi+4h}G_{\mu\nu\rho}G^{\mu\nu\rho} - \frac{1}{4}e^{-\sqrt{2}\phi+2h}(F_{\mu\nu}F^{\mu\nu} + m^2B_{\mu\nu}B^{\mu\nu}) \right. \\ & - (\nabla h)^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}e^{2\sqrt{2}\phi-4h}(\nabla B_0)^2 + \frac{1}{2}e^{-4h} + e^{-2h}P - \frac{1}{2}m^2e^{-\sqrt{2}\phi-6h}B_0^2 \\ & \left. - \frac{1}{2}\gamma^2e^{-\sqrt{2}\phi-6h} \right] + \frac{1}{8} \int d^4x (m^2B_0B \wedge B + B_0F \wedge F + \gamma B \wedge F), \end{aligned} \quad (37)$$

where $G = dB$ and the dilaton potential $P = \frac{1}{8}(g^2e^{\sqrt{2}\phi} + 4gme^{-\sqrt{2}\phi} - m^2e^{-3\sqrt{2}\phi})$.

One advantage of the consistent truncation is that it makes the relation between seemingly different solutions evident. In [13], it was noted that theories with seemingly different values of g and m were in fact related by a simple scaling of the fields, so that for $g > 0, m > 0$ the inequivalent theories were labelled by $\frac{g}{m}$. In our truncated theory, there are apparently three parameters, g, m and γ ; however, as one might expect from the preceding analysis of the fixed points, the inequivalent theories are labelled by a single invariant combination, $g^2\gamma^2$. This can be seen explicitly by starting from the theory with any $g > 0, m > 0$, setting $\gamma = \frac{\Gamma}{g}$, and making the field redefinitions

$$e^{-\sqrt{2}\phi} = \sqrt{\frac{g}{m}}e^{-\sqrt{2}\phi'}, \quad e^{-2h} = \sqrt{g^3m}e^{-2h'}, \quad B_0 = \frac{1}{gm}B'_0, \quad (38)$$

$$g_{\mu\nu} = \frac{1}{g^3m}g'_{\mu\nu}, \quad F^{(3)} = \frac{1}{g}F^{(3)'}, \quad B = \frac{1}{mg}B'. \quad (39)$$

The action (37) then reduces to an overall factor of $\frac{1}{g^3m}$ times the same action for the primed fields with $g = m = 1$, depending only on Γ . Thus, the inequivalent theories are labelled by $\Gamma^2 = g^2\gamma^2$.

Having this four-dimensional action also makes it easier to compare to the phenomenological models which have previously been studied. We can see that the action is qualitatively similar to the theory considered in [5]. The most significant difference is perhaps the presence of a mass term for the two-form B , although there are also additional scalar fields, and some more complicated couplings.

5 Flows in the $F(4)$ gauged supergravity

We wish to construct domain wall solutions which interpolate between these solutions. Assuming the fields are only functions of the radial coordinate and that the interpolating solutions preserve the rotational symmetry in the spatial directions, we can by choice of gauge write the most general such solution as

$$ds^2 = -e^{2F(r)} dt^2 + r^2 d\vec{x}^2 + e^{2d(r)} \frac{dr^2}{r^2} + e^{2h(r)} d\Omega_2^2, \quad (40)$$

where $d\Omega_2^2 = \frac{1}{y_2^2}(dy_1^2 + dy_2^2)$, with the matter fields $\phi(r)$,

$$F^{(3)} = \alpha(r) \frac{e^{F(r)+d(r)}}{r} dt \wedge dr + \frac{\gamma}{y_2^2} dy_1 \wedge dy_2, \quad (41)$$

and

$$\mathcal{H} = mB = m \frac{\bar{\beta}(r)}{2} r^2 dx_1 \wedge dx_2. \quad (42)$$

Note $dF^{(3)} = 0$ implies γ is a constant throughout the flow.

The matter equations of motion are

$$\begin{aligned} & \frac{\sqrt{2}}{r e^{F+d+2h}} \partial_r (r^3 e^{F-B+2h} \partial_r \phi) + \frac{1}{4} (g^2 e^{\sqrt{2}\phi} - 4gme^{-\sqrt{2}\phi} + 3m^2 e^{-3\sqrt{2}\phi}) \\ & - \frac{1}{2r^2 e^{2d}} e^{2\sqrt{2}\phi} (\partial_r (r^2 \bar{\beta}))^2 + \frac{m^2}{4} e^{-\sqrt{2}\phi} \bar{\beta}^2 - e^{-\sqrt{2}\phi} \alpha^2 + e^{-\sqrt{2}\phi} \gamma^2 e^{-4h} = 0, \end{aligned} \quad (43)$$

$$\frac{r}{e^{F+d+2h}} \partial_r \left(\frac{e^{2\sqrt{2}\phi+F+2h-d}}{2r} \partial_r (r^2 \bar{\beta}) \right) = \frac{m^2}{2} e^{-\sqrt{2}\phi} \bar{\beta} + 2\alpha\gamma e^{-2h}, \quad (44)$$

and

$$\frac{1}{r e^{d+2h}} \partial_r (e^{-\sqrt{2}\phi} r^2 e^{2h} \alpha) = \frac{\gamma}{r e^{d+2h}} \partial_r (r^2 \bar{\beta}). \quad (45)$$

This last equation can be integrated to obtain

$$\alpha = \gamma \bar{\beta} e^{\sqrt{2}\phi} e^{-2h}, \quad (46)$$

where we have set a constant of integration to zero because it vanishes in the solutions we want to consider interpolating between.

Einstein's equations give

$$\begin{aligned} \frac{1}{r e^{F+d+2h}} \partial_r (r^3 e^{-d+2h} \partial_r e^F) &= P + \frac{1}{4r^2 e^{2d}} e^{2\sqrt{2}\phi} (\partial_r (r^2 \bar{\beta}))^2 \\ &+ e^{-\sqrt{2}\phi} \left(\frac{m^2 \bar{\beta}^2}{8} + \frac{3\alpha^2}{2} + \frac{\gamma^2}{2} e^{-4h} \right), \end{aligned} \quad (47)$$

$$\begin{aligned} \frac{1}{r e^{F+d+2h}} \partial_r (r^2 e^{F-d+2h}) &= P - \frac{1}{4r^2 e^{2d}} e^{2\sqrt{2}\phi} (\partial_r (r^2 \bar{\beta}))^2 \\ &+ e^{-\sqrt{2}\phi} \left(-\frac{3m^2 \bar{\beta}^2}{8} - \frac{\alpha^2}{2} + \frac{\gamma^2}{2} e^{-4h} \right), \end{aligned} \quad (48)$$

$$\begin{aligned} e^{-2h} + \frac{1}{2r e^{F+d+2h}} \partial_r (r^3 e^{F-d} \partial_r e^{2h}) &= P + \frac{1}{4r^2 e^{2d}} e^{2\sqrt{2}\phi} (\partial_r (r^2 \bar{\beta}))^2 \\ &+ e^{-\sqrt{2}\phi} \left(\frac{m^2 \bar{\beta}^2}{8} - \frac{\alpha^2}{2} - \frac{3\gamma^2}{2} e^{-4h} \right), \end{aligned} \quad (49)$$

and

$$\begin{aligned} e^{-2h} + \frac{1}{f e^{2d}} (f + f r^2 (\partial_r h)^2 + 2r^2 \partial_r h \partial_r f + 2r \partial_r f + 4r f \partial_r h) \\ = r^2 e^{-2d} (\partial_r \phi)^2 + 2P + \frac{1}{4r^2 e^{2d}} e^{2\sqrt{2}\phi} (\partial_r (r^2 \bar{\beta}))^2 + e^{-\sqrt{2}\phi} \left(-\frac{m^2 \bar{\beta}^2}{4} - \alpha^2 - \gamma^2 e^{-4h} \right), \end{aligned} \quad (50)$$

where as before $P = \frac{1}{8}(g^2 e^{\sqrt{2}\phi} + 4gm e^{-\sqrt{2}\phi} - m^2 e^{-3\sqrt{2}\phi})$. The final equation comes from the rr component of the Einstein tensor rather than the Ricci tensor.

There are seven equations and only six unknown functions, but one of the equations is redundant because of the Bianchi identity, which involves the r derivative of the last equation.

As explained in the previous section, for the Lifshitz and AdS solutions, the parameters g, m in the Lagrangian for our supergravity theory only affect the overall scale of the fields; the value of z is determined entirely by $g^2 \gamma^2$. In the four-dimensional truncated theory, this was manifest at the level of the action. We will use a similar scaling of the fields here to simplify the equations of motion. However, in this context we find it convenient to include a factor of γ in the scaling of h . We set

$$\varphi = \sqrt{\frac{m}{g}} e^{-\sqrt{2}\phi}, \quad e^{-2H} = \frac{\gamma}{\sqrt{gm}} e^{-2h}, \quad e^{-2D} = \frac{1}{\sqrt{g^3 m}} e^{-2d}, \quad \beta = \sqrt{\frac{m}{g}} \bar{\beta}. \quad (51)$$

Then writing the radial coordinate as $r = e^\rho$, we have four second-order equations,

$$\begin{aligned} \partial_\rho \partial_\rho \beta = & -2\partial_\rho \beta - (2\beta + \partial_\rho \beta) (\partial_\rho F - \partial_\rho D + 2\partial_\rho H - 2\varphi^{-1} \partial_\rho \varphi) \\ & + e^{2D} \varphi (\varphi^2 + e^{-4H}) \beta \end{aligned} \quad (52)$$

$$\begin{aligned} \partial_\rho \partial_\rho \varphi = & -\partial_\rho \varphi (2 + \partial_\rho F - \partial_\rho D + 2\partial_\rho H - \varphi^{-1} \partial_\rho \varphi) - \frac{1}{2\varphi} (\partial_\rho \beta + 2\beta)^2 \\ & + \frac{e^{2D}}{4} (1 - 4\varphi^2 + 3\varphi^4 + (\varphi^2 - 4e^{-4H}) \beta^2 + 4\varphi^2 e^{-4H}) \end{aligned} \quad (53)$$

$$\begin{aligned} \partial_\rho \partial_\rho F = & \partial_\rho F (2 + \partial_\rho F - \partial_\rho D + 2\partial_\rho H) + \frac{1}{4\varphi} (\partial_\rho \beta + 2\beta)^2 \\ & + \frac{e^{2D}}{8\varphi} (1 + 4\varphi^2 - \varphi^4 + (\varphi^2 + 12e^{-4H}) \beta^2 + 4\varphi^2 e^{-4H}) \end{aligned} \quad (54)$$

$$\begin{aligned} \partial_\rho \partial_\rho H = & \partial_\rho H (2 + \partial_\rho F + \partial_\rho D - \partial_\rho H) + 2\partial_\rho H + 2\partial_\rho F + 1 \\ & + \frac{e^{2D}}{8\varphi} (1 + 4\varphi^2 - \varphi^4 + (3\varphi^2 + 7e^{-4H}) \beta^2) \end{aligned} \quad (55)$$

and two further equations

$$2 + 2\partial_\rho H + \partial_\rho F - \partial_\rho D + \frac{1}{4\varphi} (\partial_\rho \beta + 2\beta)^2 = \frac{e^{2D}}{8\varphi} (1 + 4\varphi^2 - \varphi^4 - (3\varphi^2 + 4e^{-4H}) \beta^2 + 4\varphi^2 e^{-4H}), \quad (56)$$

and

$$\begin{aligned} \frac{1}{g^2 \gamma^2} = & -\frac{e^{2H}}{4\varphi} (\varphi^2 + 4e^{-4H}) \beta^2 - \varphi e^{-2H} \\ & - e^{2H-2D} \left(1 + (\partial_\rho H)^2 + 2\partial_\rho H \partial_\rho F + 2\partial_\rho F + 4\partial_\rho H - \frac{(\partial_\rho \varphi)^2}{2\varphi^2} + \frac{(\partial_\rho \beta + 2\beta)^2}{4\varphi^2} \right). \end{aligned} \quad (57)$$

Note that this last equation involves only $g^2 \gamma^2$, while the other equations do not involve the parameters in the theory at all. Note also that the equations involve only $\partial_\rho F$, and not F .

We can look at this system in two different ways. If we consider it as a dynamical system, it is convenient to view (52-56) as a system of nine coupled first-order equations in the variables $e^H, e^D, \varphi, \beta, \partial_\rho H, \partial_\rho D, \partial_\rho \varphi, \partial_\rho \beta$ and $\partial_\rho F$. We can solve (56) algebraically for one of the variables. The system is parameter-free, and the remaining set of first-order ODEs defines an eight-dimensional autonomous dynamical system. The equation (57) then specifies a subspace in this dynamical system determined by the value of $g^2 \gamma^2$. The fact that the equations are compatible implies that this is an invariant subspace; flows starting in a space of a given value of $g^2 \gamma^2$ will remain in that space. That is, the right-hand side of (57) is constant by virtue of (52-56), as required physically for conservation of the flux on the compact space. This is a convenient way to describe the equations of motion which makes the abstract structure clear, but the high dimension unfortunately makes any detailed analysis of the structure of the flows difficult.

Alternatively, when we solve these equations explicitly to find the interpolating solutions of interest, we will specify a value of $g^2\gamma^2$ and explicitly solve (52-55) and (57). The remaining equation (56) is then redundant; it follows from (52-55) and the derivative of (57).

5.1 Linearised equations

To see which directions we would expect flows in, we consider the linearisation of the equations of motion about each of the background solutions. We first consider the linearization about the AdS_4 solutions, as this can be done analytically; we then discuss the numerical results from linearisation around the Lifshitz solutions. In the linearised equations, we find that there are some linearised modes which violate the Breitenlohner-Freedman bound [23], implying that some of these solutions are unstable.

The AdS_4 solutions have $F = \ln r$, $\beta = 0$, and φ , D and H taking the constant values given in (30,31). We linearise by writing

$$F = \ln r + \delta F, \quad D = D_0 + \delta D, \quad H = H_0 + \delta H, \quad \varphi = \varphi_0(1 + \delta\varphi), \quad (58)$$

and noting that β is itself a linear perturbation. This last fact makes the linearization simple, as some equations decouple.

There are two decoupled equations in the AdS_4 case. First a combination of (54) and (56) gives a simple decoupled equation for δF ,

$$\partial_\rho \partial_\rho \delta F + 3\partial_\rho \delta F = 0, \quad (59)$$

with solution $\delta F = f_0 + f_1 e^{-3\rho}$. As in the massive vector case, f_0 corresponds to a deformation of the timelike component of the background metric, which acts as a source for the energy density. The mode f_1 will then correspond to the vacuum expectation value of the energy density. The equation of motion (57) implies that the mode f_1 also appears in δD , corresponding to the pressure required by tracelessness of the stress tensor. Since the energy density is a precisely marginal operator, deforming by its source by turning on non-zero f_0 does not generate a flow. In fact, this is just a diffeomorphism of the bulk spacetime, rescaling the time coordinate.

We also get a simple decoupled linear equation for β from (52),

$$\partial_\rho \partial_\rho \beta = -3\partial_\rho \beta - 2\beta + e^{2D_0} \varphi_0 (\varphi_0^2 + e^{-4H_0}) \beta. \quad (60)$$

If we look for solutions of the form $\beta = \beta_1 e^{-\Delta\rho}$, this implies

$$(\Delta - 2)(\Delta - 1) = -6 \frac{2\varphi_0^4 - 4\varphi_0^2 + 1}{\varphi_0^2(2 - \varphi_0^2)}, \quad (61)$$

where φ_0 takes one of the two possible values given by solving (30). Since β appears quadratically in the other equations of motion, this solution will not source other fields at linear order.

There are two solutions $\Delta_1 > \Delta_2$ such that $\Delta_1 + \Delta_2 = 3$. With standard boundary conditions, these solutions correspond to the expectation value of and source for a dual operator of conformal dimension $\Delta = \Delta_1$. When the dual operator is relevant, turning on the source will deform the solution, generating a flow starting from this solution in the UV.

The operator is relevant, $\Delta < 3$, if $\varphi_0^2 < 0.3675$. This is in the range corresponding to the small φ_0 solution, and corresponds to $g^2\gamma^2 < 0.23$, when there is a single Lifshitz solution. Since turning on this mode corresponds to deforming the solution by exciting the two-form which is present in the Lifshitz solution but absent in the AdS solution, the natural guess is that this will lead to an RG flow from the small φ_0 AdS solution in the UV to the Lifshitz solution in the IR. When $\varphi_0^2 > 0.3675$, the operator is irrelevant, and we would expect to have flows that approach the AdS solution in the IR along this direction. We will show that flows from the smaller z Lifshitz solution in the UV can indeed reach these solutions in the IR.

As noted above, our system of equations can be reduced to an eight-dimensional dynamical system. There will therefore be four more linearly independent solutions of the linearised equations of motion. We can obtain the associated powers by considering (53) and (55), which give

$$\frac{2}{3}\varphi_0^2(2-\varphi_0^2)(\partial_\rho\partial_\rho\delta\varphi + 3\partial_\rho\delta\varphi) = -2(1-3\varphi_0^4)\delta\varphi + 4(1-3\varphi_0^2)(1-\varphi_0^2)\delta H \quad (62)$$

and

$$\frac{4}{3}\varphi_0^2(2-\varphi_0^2)(\partial_\rho\partial_\rho\delta H + 3\partial_\rho\delta H) = 2(1-3\varphi_0^2)(1-\varphi_0^2)\delta\varphi + 4(-1+8\varphi_0^2-5\varphi_0^4)\delta H. \quad (63)$$

These linear perturbations are dual to a pair of scalar operators in the dual CFT. As the equations are coupled, we should perform a field redefinition to diagonalise this system to obtain the bulk fields dual to the individual operators. But as all we are mainly interested in is finding the dimensions of the operators, we can proceed by considering a solution of the form $\delta\varphi = \varphi_1 e^{-\Delta\rho}$, $\delta H = H_1 e^{-\Delta\rho}$, which gives

$$[\varphi_0^2(2-\varphi_0^2)\Delta(\Delta-3) - 3(3\varphi_0^4-1)]\varphi_1 = 6(1-4\varphi_0^2+3\varphi_0^4)H_1, \quad (64)$$

and

$$[\varphi_0^2(2-\varphi_0^2)\Delta(\Delta-3) + 3(1-8\varphi_0^2+5\varphi_0^4)]H_1 = \frac{3}{2}(1-4\varphi_0^2+3\varphi_0^4)\phi_1. \quad (65)$$

Solving for Δ ,

$$\varphi_0^2(2-\varphi_0^2)\Delta(\Delta-3) = -(6\varphi_0^4-15\varphi_0^2+\frac{9}{2}) \pm \frac{3}{2}(\varphi_0^2-1)\sqrt{25\varphi_0^4-6\varphi_0^2+1}. \quad (66)$$

As before, we get solutions in pairs with $\Delta_1 > \Delta_2$ such that $\Delta_1 + \Delta_2 = 3$. With standard boundary conditions, these correspond to the expectation value of and source for a dual scalar operator of conformal dimension $\Delta = \Delta_1$. The dimensions of the two scalar operators can thus be obtained by taking the larger solution for Δ in (66) for each choice of sign.

For the upper sign, the solutions of (66) are real and the operator has $\Delta > 3$ for all $\varphi_0^2 \in (\frac{1}{3}, 1)$. This thus corresponds to an irrelevant operator around the AdS solution, and we would expect it to correspond to a direction along which we can approach either of these AdS solutions. We will see below that this direction can be reached by flows from the asymptotically AdS₆ solution in the UV (we construct the flow to the smaller φ_0 solution, but we expect such a flow to exist also for the larger φ_0 solution).

For the lower sign, the solutions of (66) are complex for $\varphi_0^2 < 0.354$. This indicates that the linearised scalar in the bulk has a mass violating the Breitenlohner-Freedman bound. Here we have restricted to an ansatz where the fields have only radial dependence, but one can easily extend this to analyse general linearised perturbations around the AdS solution in the context of the four-dimensional action (37) to see this explicitly. There is a decoupled pair of equations involving the perturbations $\delta h(x^\mu)$ and $\delta \phi(x^\mu)$. Diagonalizing these equations gives two massive scalars on the AdS background, with masses

$$m^2 L^2 = -\frac{3}{\varphi_0^2(2 - \varphi_0^2)}[(1 - 4\varphi_0^2 + \varphi_0^4) \pm (1 - \varphi_0^2)\sqrt{25\varphi_0^4 - 6\varphi_0^2 + 1}], \quad (67)$$

where L^2 is the background AdS scale. The Breitenlohner-Freedman bound is violated if $m^2 L^2 < -\frac{9}{4}$, which happens for the upper sign for sufficiently small φ_0 . Thus, this mode indicates a dynamical stability of the AdS₄ spacetime to exponentially growing modes for this scalar. At $\varphi_0 = \frac{1}{3}$, $\gamma = 0$, and this mode is simply an excitation of δh , the overall volume of the compact space. As we increase the flux, moving away from $\varphi_0 = \frac{1}{3}$, the unstable mode involves excitation of the dilaton as well.

Setting aside this issue of instability, we can look at the issue of relevant operators. The operator has $\Delta < 3$ for $\varphi_0^2 < 0.59$. This is the mid-point corresponding to $g^2\gamma^2 = 1.18$, so the AdS solution with smaller φ_0 has a relevant mode which excites the scalars but not the two-form. The natural interpretation is that a deformation by this operator will lead to an RG flow from the AdS solution with smaller φ_0 in the UV to the AdS solution with larger φ_0 in the IR. We will construct such solutions explicitly in the next section.

In summary, there are four operators which appear in our analysis around AdS; the field theory energy density, which is marginal; the operator dual to the excitation β of the two-form, which is relevant for $\varphi_0^2 < 0.3675$, and two scalar operators, one of which is always irrelevant, the other of which is relevant for $\varphi_0^2 < 0.59$. The dimensions of these operators as a function of φ_0 are shown in figure 7.

We now turn to the linearisation around the Lifshitz solutions. These are more complicated, as nothing obviously decouples. We can however rewrite the problem as a simple linear analysis problem by writing the linearised system of equations as $r \frac{d}{dr} \delta \mathbf{x} = A \delta \mathbf{x}$, where $\delta \mathbf{x}$ collectively denotes the linearised perturbations, and A is a matrix depending on the background field values. The problem of finding solutions then reduces to finding the eigenvectors and eigenvalues of the matrix A . If A has eigenvectors \mathbf{v}_i with eigenvalues Δ_i , the linearised system has solutions $\delta \mathbf{x} = \sum_i \mathbf{v}_i e^{-\Delta_i \rho}$. The eigenvalues were calculated numerically.

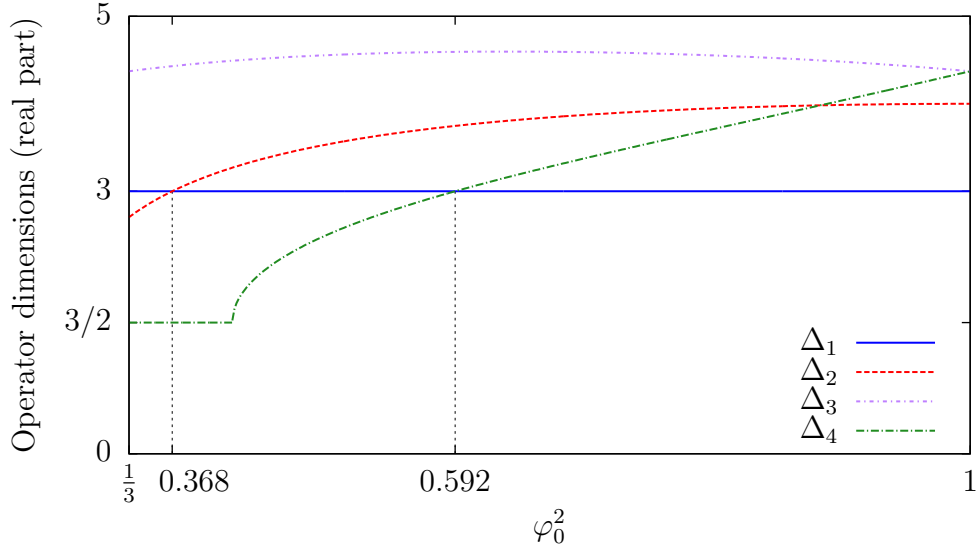


Figure 7: Operator dimensions for the linearised perturbations about the AdS solutions, determined from the asymptotic scaling of bulk fields. Δ_1 is the energy density, Δ_2 is the mode which excites the two-form, and Δ_3, Δ_4 are the modes which excite the scalars h, ϕ . We plot the real parts, but note that Δ_4 is complex where the real part is $\frac{3}{2}$, and it will then not correspond to the dimension of an operator.

In the larger z solutions, there is a pair of complex eigenvalues with real part $\frac{z+2}{2}$ for $z < 16.82$. Extrapolation from the AdS case would lead us to expect that this is associated with a dynamical instability of the Lifshitz solution, although constructing such an instability explicitly requires work outside of our ansatz. For the smaller z solutions, the situation is a little more complicated. There is a similar pair of complex eigenvalues for $z < 5.69$, then there is a small window where all eigenvalues are real up to $z = 5.83$, and then a set of four complex eigenvalues but whose real parts are not $\frac{z+2}{2}$. We would again expect that, at least for $z < 5.69$, the complex eigenvalues signal a mode which is violating the Breitenlohner-Freedman bound. In the second region of complex eigenvalues, the interpretation is less clear, but it is certainly problematic to interpret these eigenvalues in terms of operator dimensions in a dual field theory.

This instability is clearly an important aspect of the physics of these Lifshitz solutions, particularly as it is appearing for phenomenologically interesting values like $z = 2$ which we would most like to understand. An important task for the future will be to perform a more general linearised analysis to exhibit the instability explicitly and understand its character.

For the present, we will leave this instability to one side and return to analysing the flows. We have not considered in detail the identification of these linearised solutions with dual operators, so the interpretation from the field theory point of view is somewhat heuristic, but we see the structures we would expect.⁴ As in the AdS case,

⁴Extending [28] to construct the holographic dictionary for this theory should in principle be straightforward.

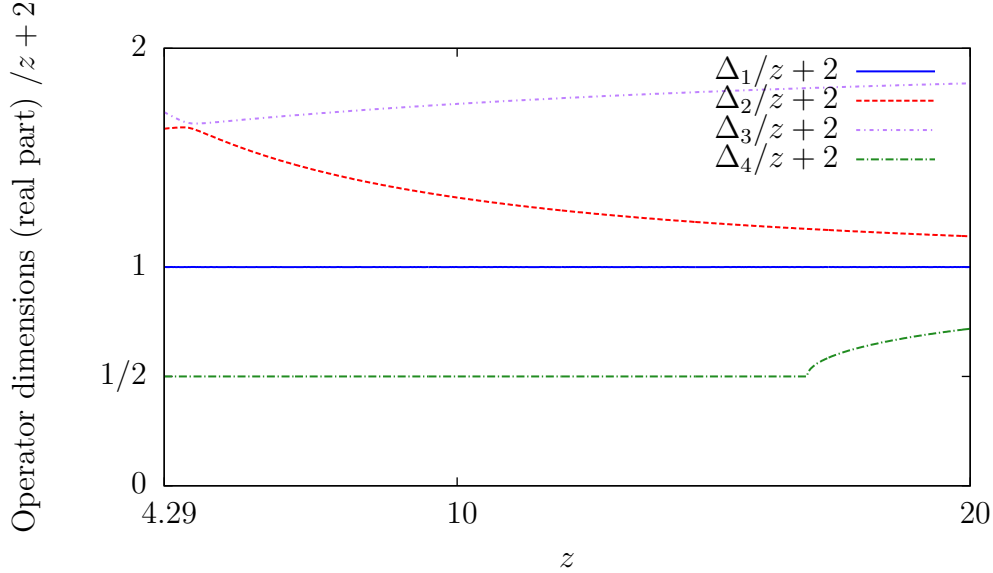


Figure 8: Real part of the scaling of the linearised perturbations about the larger z Lifshitz solution. The values are normalized to $z + 2$. When the values are real, they should be interpreted as the dimension of the corresponding operator, assuming the usual holographic dictionary. In particular, Δ_1 corresponds to the energy density. Note that Δ_4 is complex in the region where the real part is $\frac{z+2}{2}$.

the solutions come in pairs with $\Delta_i + \Delta_{i+1} = z + 2$, which should correspond to the source for and expectation value of dual operators respectively. When $\Delta_i > z + 2$, $\Delta_{i+1} < 0$, so the effect of this “source mode” is large at large r , and corresponds to a deformation which grows in the UV; conversely if $\Delta_i < z + 2$, the effect is important at small r , corresponding to a deformation which grows in the IR. The operator dimensions (the larger eigenvalue in each pair) are plotted in figures 8 and 9.

To construct interpolating solutions dual to renormalization group flows, we are therefore interested in the perturbation of the Lifshitz solution by these “source modes”. Since our explicit construction works out from the IR by considering a perturbation along an irrelevant direction, this construction is insensitive to the existence of the complex eigenvalues, at least at linear order. Thus, we construct interpolating solutions dual to flows without considering whether the solutions we are interpolating between are stable.

Of the four deformations included in our ansatz, there is one which is exactly marginal; this corresponds again to the energy density in the dual field theory. There is one irrelevant operator in the smaller z Lifshitz solution, and two in the larger z Lifshitz solution.

In summary, the expected flows here are very similar to in the massive vector model. There should be a flow from the small φ_0 AdS solution to the large φ_0 AdS solution. In the regime where there is a single Lifshitz solution, there should be flow from the small φ_0 AdS solution in the UV to the larger z Lifshitz solution in the IR. Once the smaller z Lifshitz solution appears, this will be replaced by two flows

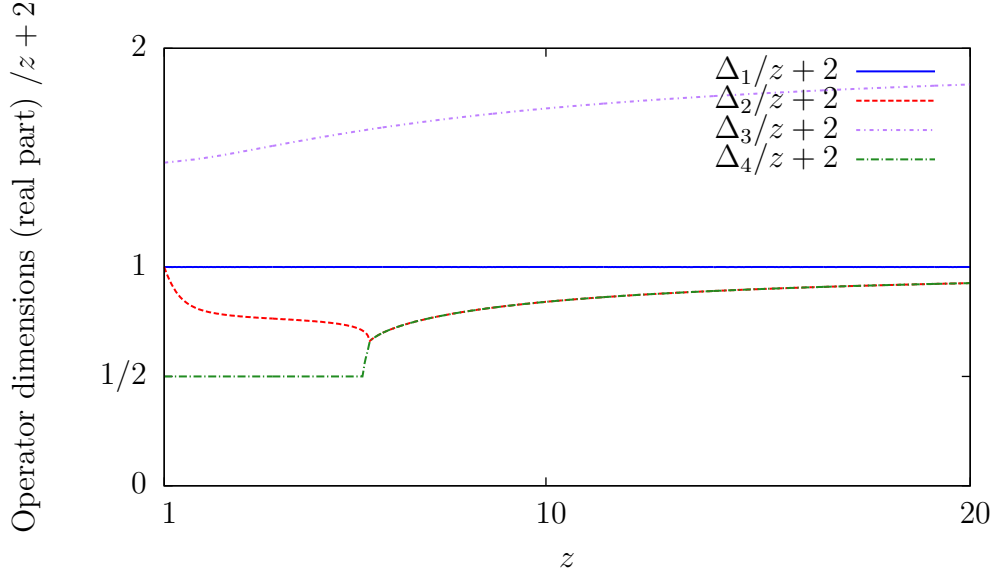


Figure 9: Real part of the scaling for the linearised perturbations about the smaller z Lifshitz solution. The values are normalized to $z + 2$. When the values are real, they should be interpreted as the dimension of the corresponding operator, assuming the usual holographic dictionary. In particular, Δ_1 corresponds to the energy density. Note that Δ_4 is complex in the region where the real part is $\frac{z+2}{2}$, and that Δ_2 and Δ_4 are both complex after their real parts merge.

starting from the smaller z Lifshitz solution in the UV, and running to the larger z Lifshitz solution or an AdS solution in the IR. In addition to the ones required by these expected flows, there is an additional irrelevant direction about each solution; we will see in the next subsection that this is associated with a flow from an asymptotically AdS_6 solution.

5.2 Flows

We can now turn to the discussion of the numerical solutions interpolating between the Lifshitz and AdS solutions. As in the massive vector model, solutions are obtained by starting from a candidate IR fixed point at small r , perturbing along one of the eigenvectors associated with an irrelevant direction, and integrating out to large r to identify the UV fixed point at the source of the renormalization group flow. The analysis is made more complicated in the present context because of the existence of more than one irrelevant direction in many cases, which implies that to reach the desired UV fixed point we have to search for an appropriate direction for the perturbation at small r . However, with the exception of the large φ_0 AdS solutions, we have only one or two irrelevant directions, so this search can be simply carried out by interval bisection in the space of possible directions on the two-dimensional plane spanned by the two irrelevant eigenvectors.

There are a large number of cases to consider, so we have relegated the plots of numerical solutions to appendix A, and here give a description of the results and their

interpretation.

5.2.1 Flows from 6D AdS

The simplest case to consider is the generic perturbation from the AdS or Lifshitz solutions, perturbing along the most irrelevant direction. Here we have only a choice of sign in the perturbation. These solutions are particularly interesting as they enable us to give a description of the field theory dual to the Lifshitz solutions in the context of this gauged supergravity theory.

These flows do not approach any of the fixed points we discussed previously in the UV; they have e^{2H} scaling like r^2 in the UV, and e^{2D} tending to some finite, non-zero value (for one choice of sign). That is, the compact hyperbolic space has a proper size growing like r^2 at large r . These can therefore be identified as flows from an asymptotically AdS_6 geometry,

$$ds^2 \approx -r^2 dt^2 + r^2 (dx_1^2 + dx_2^2) + L^2 \frac{dr^2}{r^2} + C \frac{r^2}{y_2^2} (dy_1^2 + dy_2^2). \quad (68)$$

When the solution in the IR is AdS_4 , this type of flow was previously discussed in [21], building on the work of [22]. The asymptotics correspond to considering a five-dimensional field theory on $\mathbb{R}^{1,2} \times H^2$. Specifically, the field theory is the five-dimensional $\mathcal{N} = 2$ conformal field theory obtained from the IR limit of the $Sp(N)$ D4-D8 theory. The conformal symmetry in the UV is broken by introducing curvature in the background spacetime, giving an asymptotically AdS_6 geometry.⁵ In the UV, the presence of the flux γ on the compact space implies that the UV field theory is a twisted field theory, as in [22]. The interpolating geometries with AdS_4 in the IR then describe the flow from the a five-dimensional field theory in the UV to a three-dimensional conformal theory in the IR. If we choose $\varphi_0^2 = \frac{1}{2}$, which corresponds to $g^2 \gamma^2 = 1$, the AdS_4 solution preserves half the supersymmetry. This supersymmetry is in fact preserved along the flow. It is these supersymmetry-preserving flows which are explicitly considered in [21].

To understand the flows to the Lifshitz theories in the IR, we just need to consider a more general deformation of the asymptotically AdS_6 solution where we turn on the two- form by considering a perturbation involving β . The field theory dual to the Lifshitz solutions can then be defined as the result of considering the IR limit of the $\mathcal{N} = 2$ twisted field theory on $\mathbb{R}^{1,2} \times H^2$ with this further deformation. In principle, this gives a constructive definition of the field theory duals of the Lifshitz solutions of the gauged supergravity theory, in terms of a controllable deformation of an explicit supersymmetric field theory. It would be interesting to understand this description further. Examples of such flows with either an AdS or a Lifshitz solution in the IR are shown in figures 10 and 11.

⁵This is thus technically slightly different from the flows constructed by deforming the field theory Lagrangian we are otherwise considering, but the construction of the bulk solution is essentially the same.

5.2.2 Asymptotically AdS flows

Next we consider flows from an AdS_4 solution in the UV. The simplest case is flows between two AdS solutions; the linearised analysis led us to expect a solution interpolating between the small φ_0 and large φ_0 AdS solutions. This flow will have $\beta = 0$, so we can look for it by starting from the large φ_0 AdS solution at small r perturbed by some linear combination of the two irrelevant directions associated with the scalar operators, which keep us in this subspace.

To find a flow to the small φ_0 solution, we scan across the possible linear combinations. We find that the flow that hits the other AdS point is very close to the Δ_4 direction (perturbations that involve the Δ_3 direction will lead to the asymptotically AdS_6 solution considered above). A typical example of such a flow, from $\varphi_0^2 = 0.425$ to $\varphi_0^2 = 0.8$, is shown in figure 12.

In [16], it was pointed out that interpolating solutions which approach a Lifshitz solution in the UV and an AdS solution in the IR often have a mild singularity in the IR. In fact, the singularity pointed out there is a purely IR feature, which will appear whenever an interpolating solution approaches AdS in the interior, and the associated irrelevant direction in the field theory is close to marginal. We discuss this here as the AdS to AdS flows we are considering here provide a simple example of this point.

Consider the behaviour of the flow near the IR fixed point. This is controlled by the leading irrelevant direction, which for the flows considered here will be the one associated with the eigenvalue Δ_4 . The IR fields are therefore of the general form $\delta\mathbf{x} \sim r^{\Delta_4-3}$. As $r \rightarrow 0$, these perturbations decay to zero. However, as $\Delta_4 < 5$ in all the large φ_0 solutions, the second derivatives $\partial_r^2 \delta\mathbf{x}$ will blow up at small r . As noted in [16], this is reflected in a divergence of the components of the Riemann tensor in a parallelly propagated orthonormal frame as $r \rightarrow 0$. This divergence should signal some pathology in the behaviour of the field theory. Further analysing these divergences is an interesting problem for the future; these AdS to AdS flows provide a useful laboratory to do so, as they are simple deformations of a relativistic conformal field theory, although they do not preserve supersymmetry, as the large φ_0 AdS solution is never supersymmetric.

Flows from an AdS solution in the UV to a Lifshitz solution in the IR are only expected to be possible for the small φ_0 AdS solution for $g^2\gamma^2 \lesssim 0.227$, as this is when the operator dual to β is relevant. We expect the IR end of this flow to be the larger z Lifshitz solution, as it has an additional irrelevant direction (in addition to the one associated with flows from AdS_6). We numerically found such flows for a range of values of z . The flow with $z = 5$ in the IR is shown in figure 13.

5.2.3 Asymptotically Lifshitz flows

We will also have flows with a Lifshitz field theory in the UV; as discussed previously, when $g^2\gamma^2 > 0.227$, the AdS solution can no longer be the UV fixed point associated with the larger z Lifshitz solution in the IR. Since this is the value at which the smaller z Lifshitz solution appears, it is natural to assume that the flow is replaced

by new flows involving this solution. Indeed, shooting from the IR will now produce flows from the smaller z Lifshitz solution in the UV to the larger z Lifshitz solution in the IR. The flow from $z = 1.248$ to $z = 9$ is shown as figure 14.

We would also expect there to be a Lifshitz to AdS flow in this range of parameters, corresponding roughly speaking to deforming the smaller z Lifshitz solution in the opposite direction, decreasing β . We should be able to obtain the flows numerically by starting from the small φ_0 AdS solution and considering the irrelevant perturbation along the Δ_2 direction. Such flows could also exist for the large φ_0 AdS solutions, but they might be more difficult to find because of the additional irrelevant direction there. In fact, it proved possible to find flows with both AdS solutions in the IR using a simple shooting algorithm. Examples are shown in figures 15 and 16 respectively. The differences between these two flows is typical of the difference between flows from the same Lifshitz space to AdS spaces on the two different branches.

There is an interesting special case of the flow from smaller z Lifshitz in the UV to large φ_0 AdS in the IR. Since there is also a flow from the Lifshitz solution to the small φ_0 AdS solution, it should be possible to tune the deformation so that the flow from smaller z Lifshitz in the UV to large φ_0 AdS passes near the fixed point associated with the small φ_0 AdS solution at intermediate energy scales. These flows provide an interesting illustration of the general shooting technique, showing how starting from a more generic flow we can tune in to a different flow by varying the irrelevant direction of perturbation at the IR end of the flow. An illustrative flow geometry is shown in figure 17. Note that such solutions are only possible for the region of parameter space in which Δ_3 is positive for both AdS solutions, namely $0.227 \lesssim g^2\gamma^2 \lesssim 1.185$. It would also be interesting to understand what happens to the flows from the Lifshitz solution for $g^2\gamma^2 > 1.185$, when the AdS solutions no longer exist.

Since the smaller z Lifshitz branch emanates from the small φ_0 AdS solution at $g^2\gamma^2 \approx 0.23$, flows from a smaller z Lifshitz fixed point in the UV to the small φ_0 AdS solution in the IR for z near 1 involve only a small change from the initial solution. We therefore thought it might be interesting to see if such flows could be analysed perturbatively. However, it is easy to see that they cannot be analysed in a purely linear approximation: starting from the IR AdS solution, the decoupled equation (60) tells us that $\beta = \beta_1 e^{-\Delta\rho}$ at the linearised level, and the linear approximation must therefore break down at sufficiently large r , with the quadratic or higher corrections causing the asymptotic value of β to approach the constant value associated with the Lifshitz solution. Thus, even though the asymptotic value of β is small for z near 1, a simple linearised analysis of the flow is not possible.

6 Conclusions

Our main goal in this paper was to explore the renormalization group flows between theories with isotropic or anisotropic scaling symmetries from the dual holographic viewpoint. This analysis sheds further light on the field theory interpretation of the Lifshitz geometries; several motivations and possible applications were discussed in the introduction.

We studied the flows in the context of the simple massive vector model of [6] and in the gauged supergravity theory considered in [8]. We found that the two theories had a surprisingly similar structure of flows. There were a range of different possibilities: flows between different AdS solutions, AdS to Lifshitz flows, Lifshitz to AdS flows and Lifshitz to Lifshitz flows. In the gauged supergravity model, there were also flows from a six-dimensional asymptotically AdS solution to four-dimensional AdS or Lifshitz solutions. These last cases are particularly interesting as they offer a description of the field theory dual to the Lifshitz solutions, in terms of a flow from a five-dimensional $Sp(N)$ theory compactified on H^2/Γ . Exploring the consequences for the structure of the field theory dual to the Lifshitz solution is an interesting direction for future work.

In analysing the linearised perturbations, we noted that some of the AdS and Lifshitz solutions of the gauged supergravity theory appear to be unstable. This contrasts with the massive vector model, where no instability appeared within our ansatz. The appearance of instabilities for non-supersymmetric AdS solutions is not a surprise; [32] has argued that this will be generic for non-supersymmetric AdS solutions. Further analysis and characterisation of these instabilities is probably the most important open direction in our work.

Another direction for further development will be to consider a similar analysis for other theories, notably the similar construction of three-dimensional Lifshitz geometries in type IIB supergravity in [8]. In the context of the circle reductions which give $z = 2$ Lifshitz solutions, it would be interesting to study the flows in the model of [11], which has both Lifshitz and AdS solutions. One could also look for flows between higher-dimensional AdS solutions and Lifshitz solutions in all these circle reductions. It would also be interesting to further analyse the curvature singularity in the IR region of the flows signalled in [16] and to understand its interpretation in the field theory.

Acknowledgements

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A Numerical plots for $F(4)$ theory

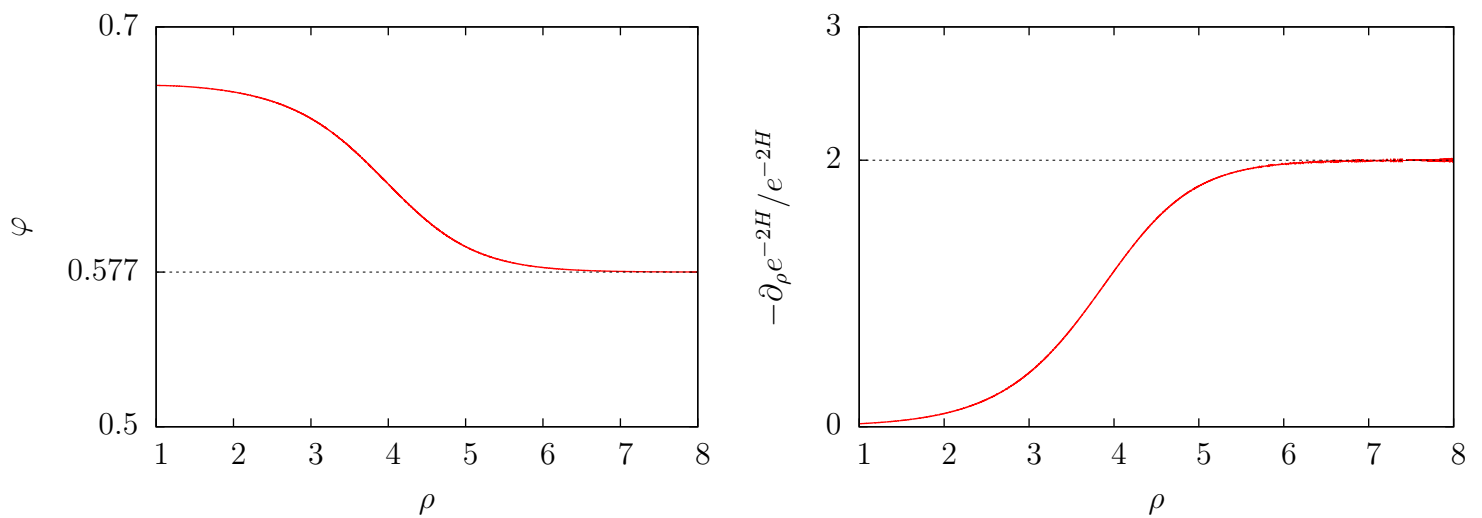


Figure 10: Holographic RG flow from the 6D spacetime (68) to the $\varphi_0^2 = 0.45$ AdS solution. $F = 1$, $\beta = 0$ throughout this flow.

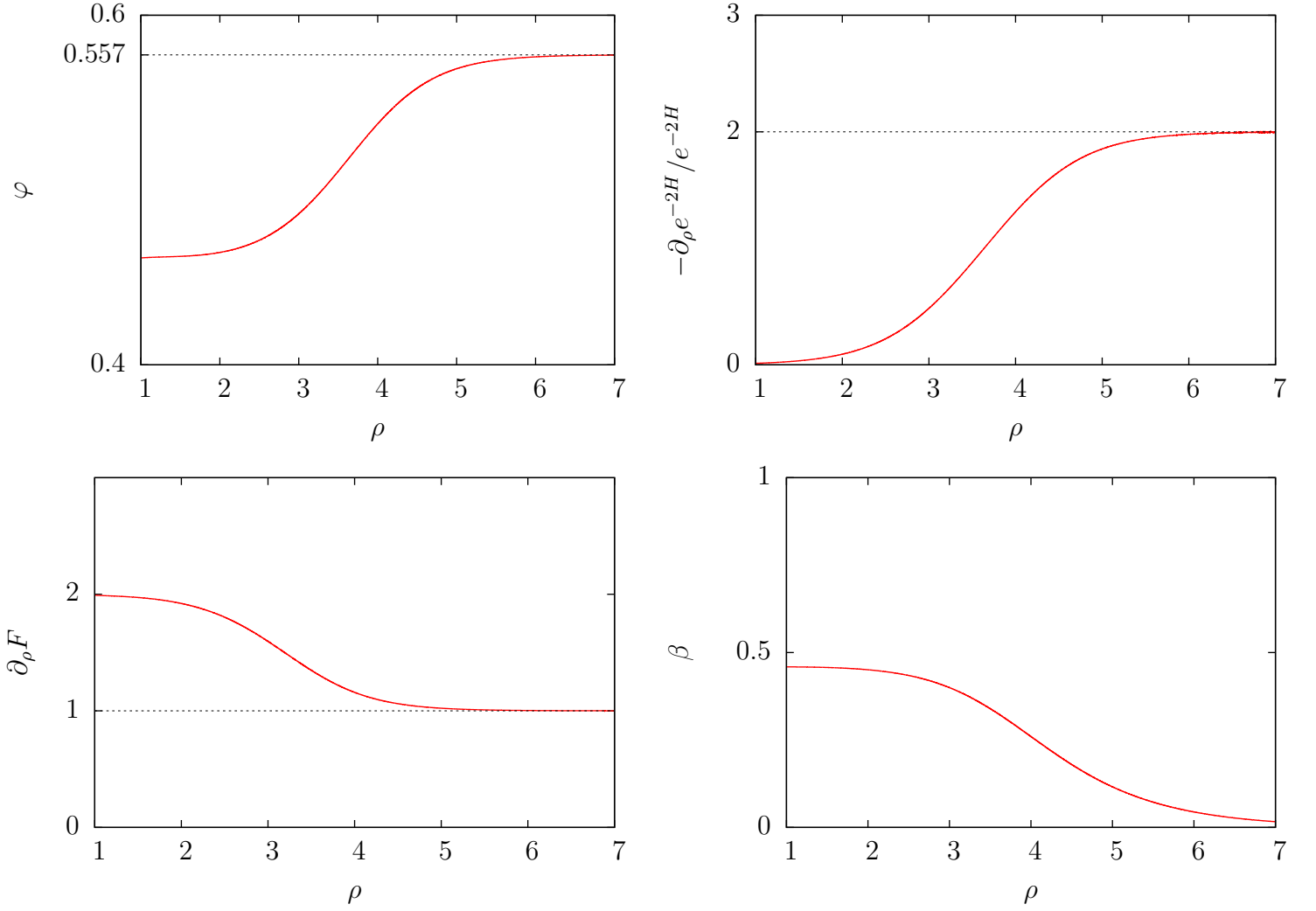


Figure 11: Holographic RG flow from the 6D spacetime (68) to the $z = 2$ Lifshitz solution on the smaller z branch of Lifshitz solutions.

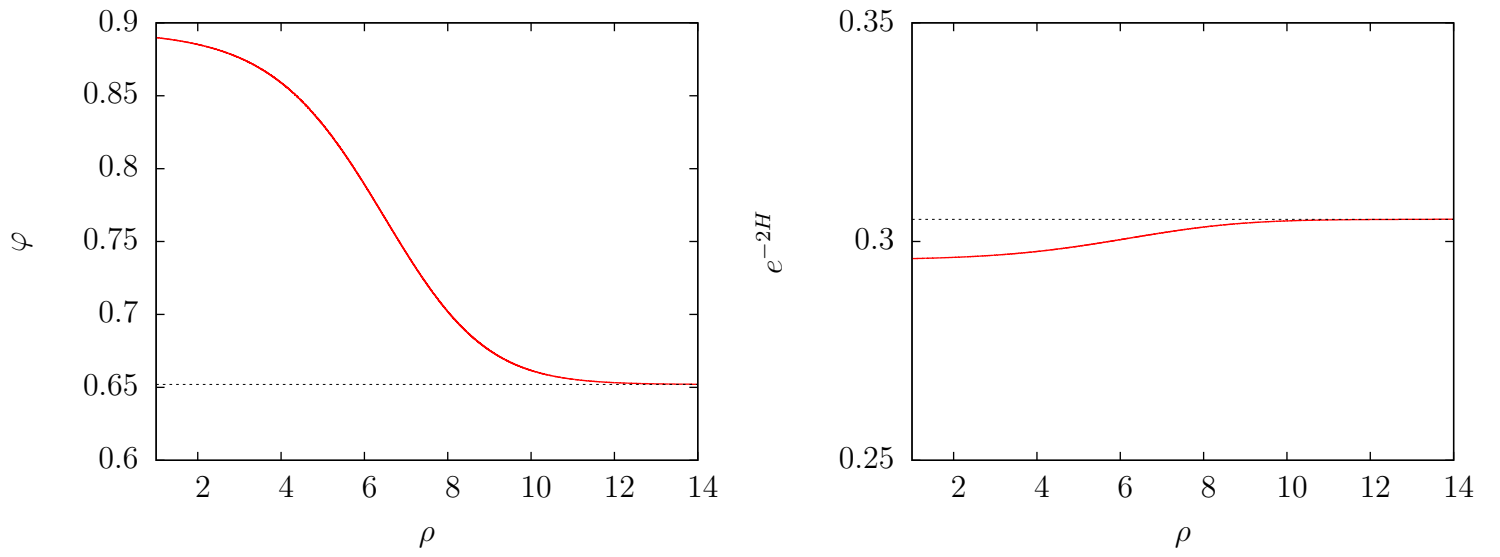


Figure 12: Holographic RG flow from an AdS space with $\varphi_0^2 = 0.425$ to an AdS space with $\varphi_0^2 = 0.8$. The dashed lines show the exact values of φ_0 and e^{-2H} of the small φ_0 AdS space we expected to hit. $f = r$ and $\beta = 0$ throughout the flow, as expected.

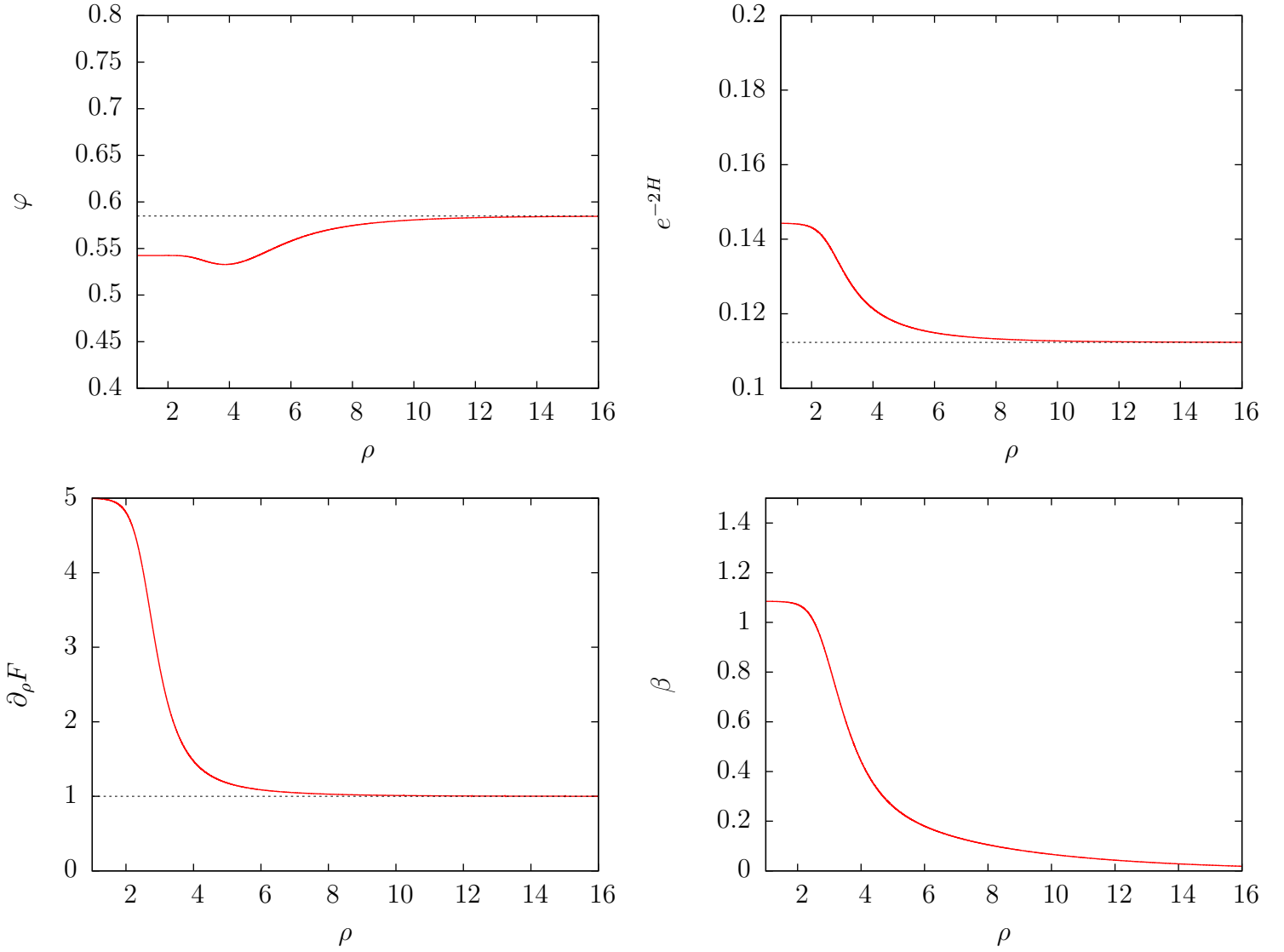


Figure 13: Holographic RG flow from an AdS space with $\varphi_0^2 = 0.342$ to a Lifshitz space on the lower sign branch with $z = 5$. The dashed lines show the exact field values of the UV AdS solution. Note that F provides an estimate of z .

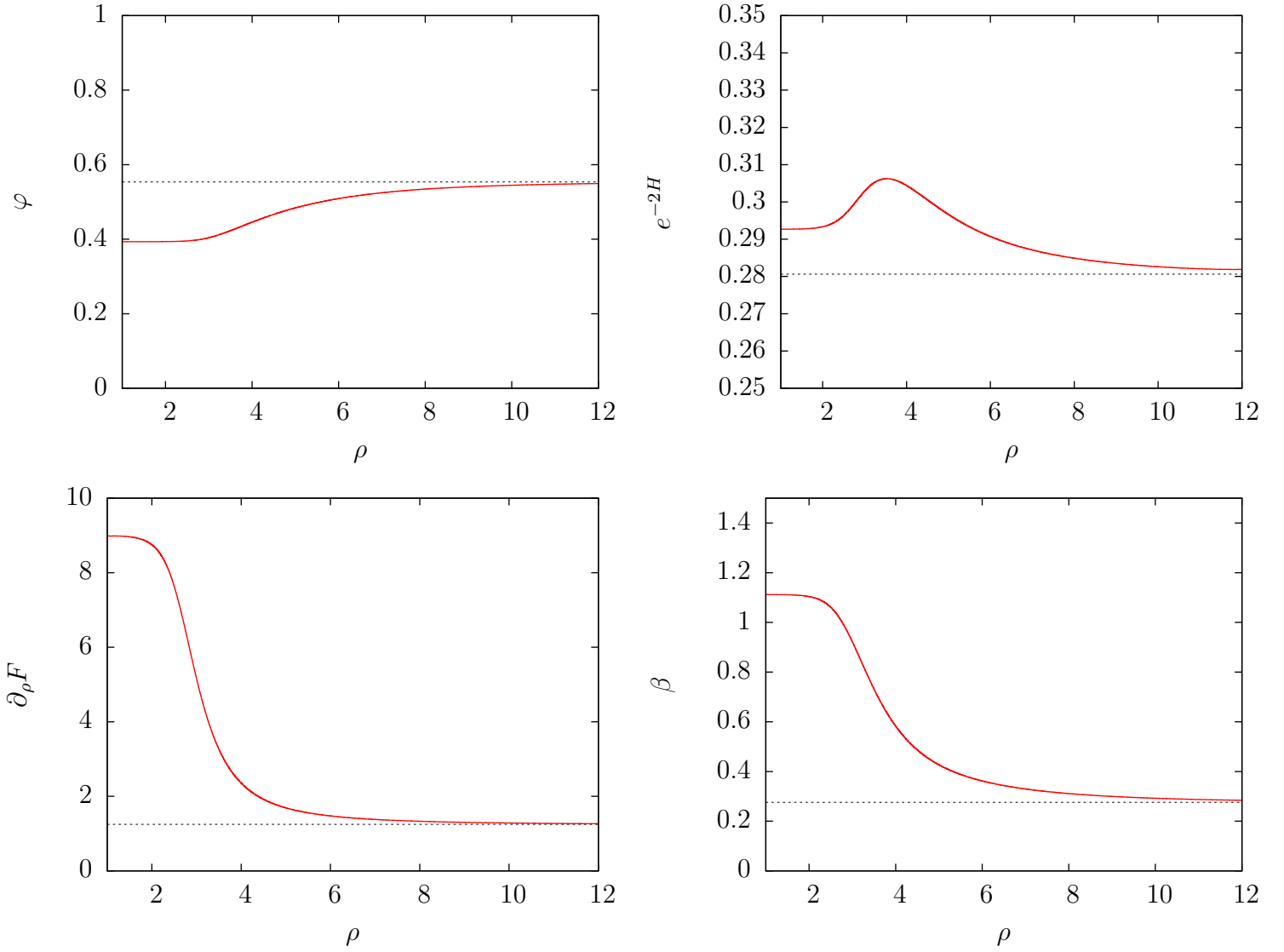


Figure 14: Holographic RG flow from an smaller z Lifshitz space with $z = 1.248$ to a larger z branch Lifshitz solutions with $z = 9$. The dashed lines show the exact field values of the Lifshitz space in the UV. Note that F provides an estimate of z .

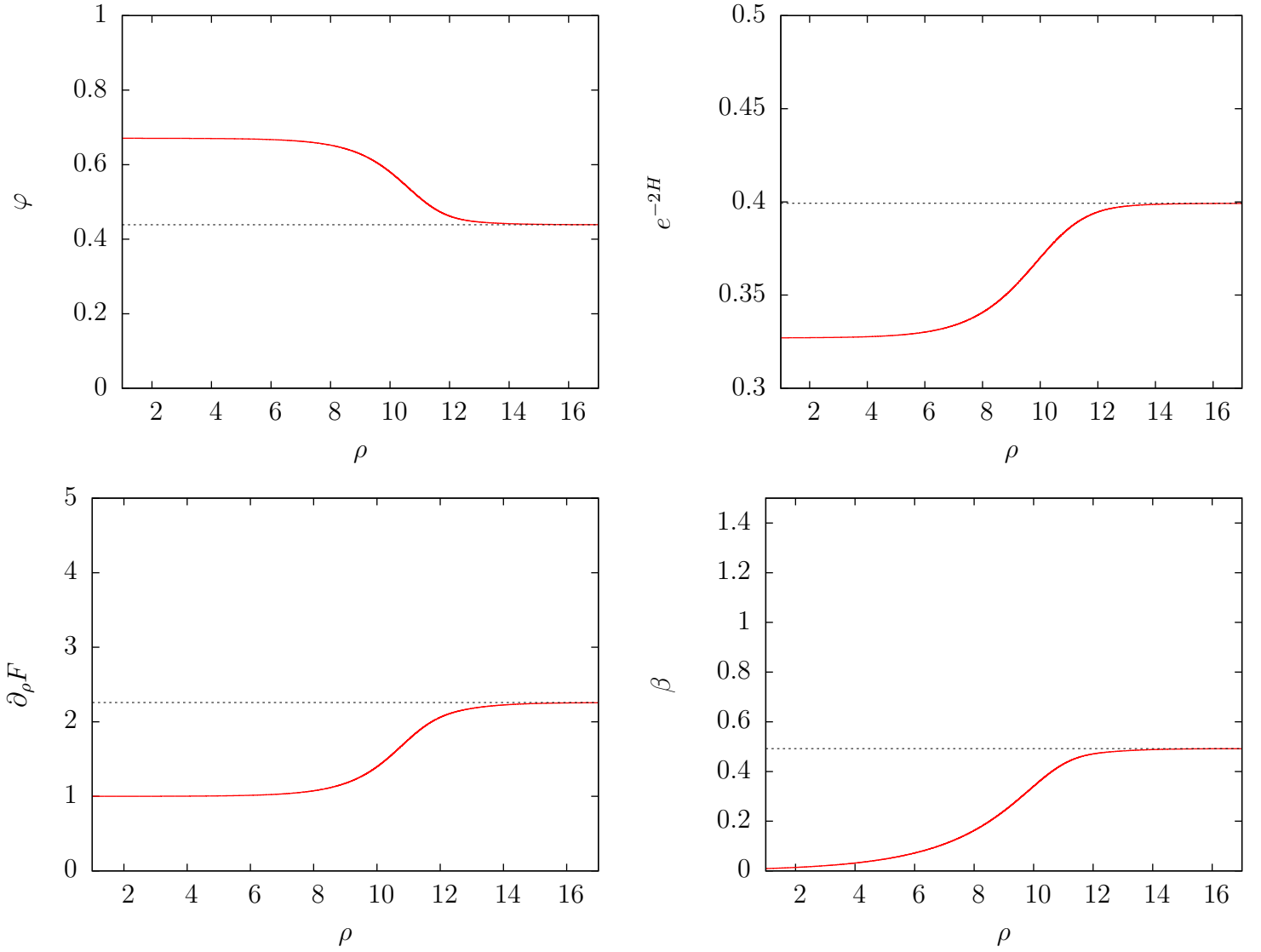


Figure 15: Holographic RG flow from an smaller z Lifshitz space with $z = 2.258$ to an AdS space on the small φ_0 branch with $\varphi_0^2 = 0.45$. The dashed lines show the exact field values of the Lifshitz space.

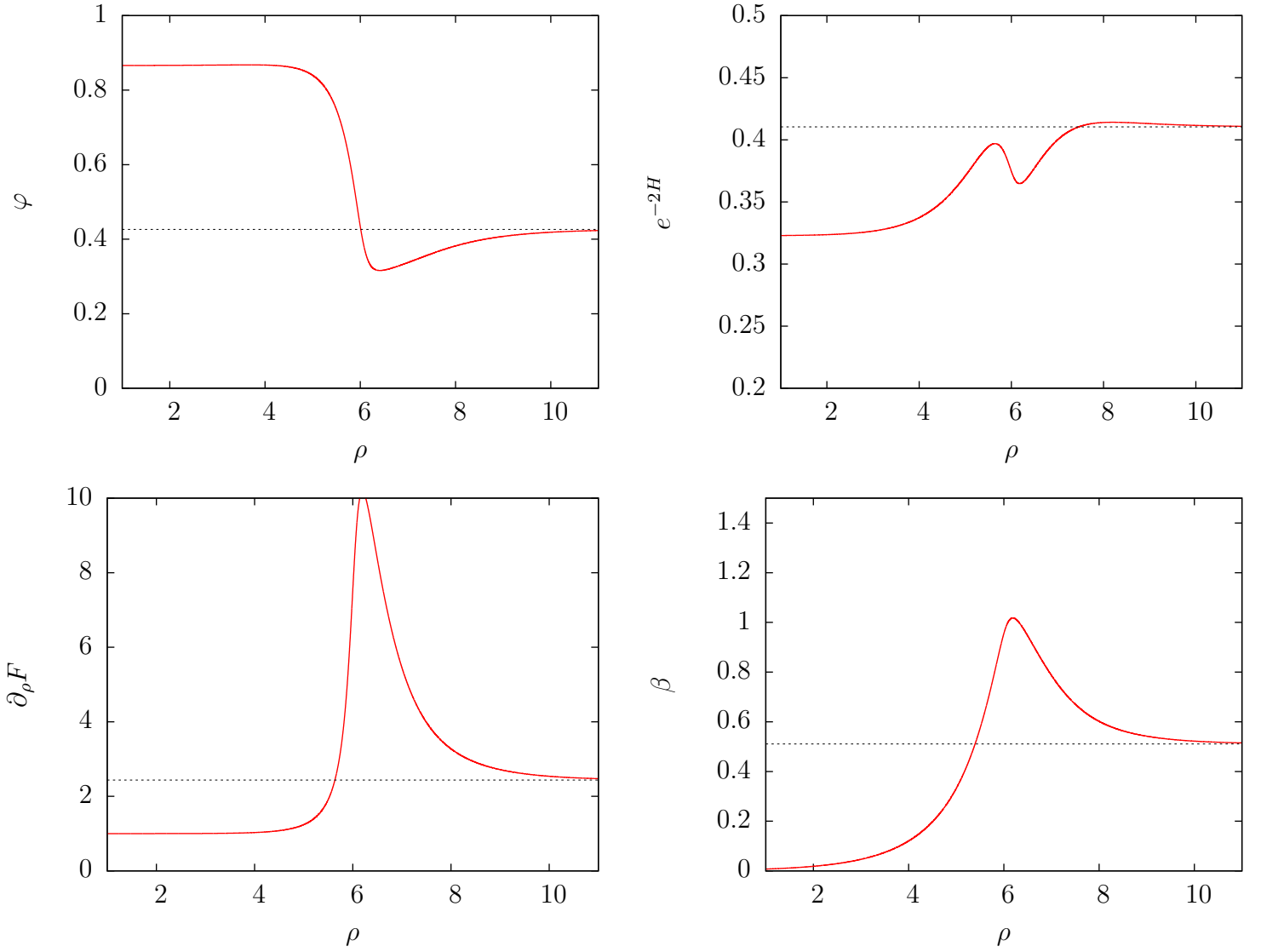


Figure 16: Holographic RG flow from a small z Lifshitz space with $z = 2.437$ to an AdS space on the large φ_0 branch with $\varphi_0^2 = 0.75$. The dashed lines show the exact field values of the Lifshitz space.

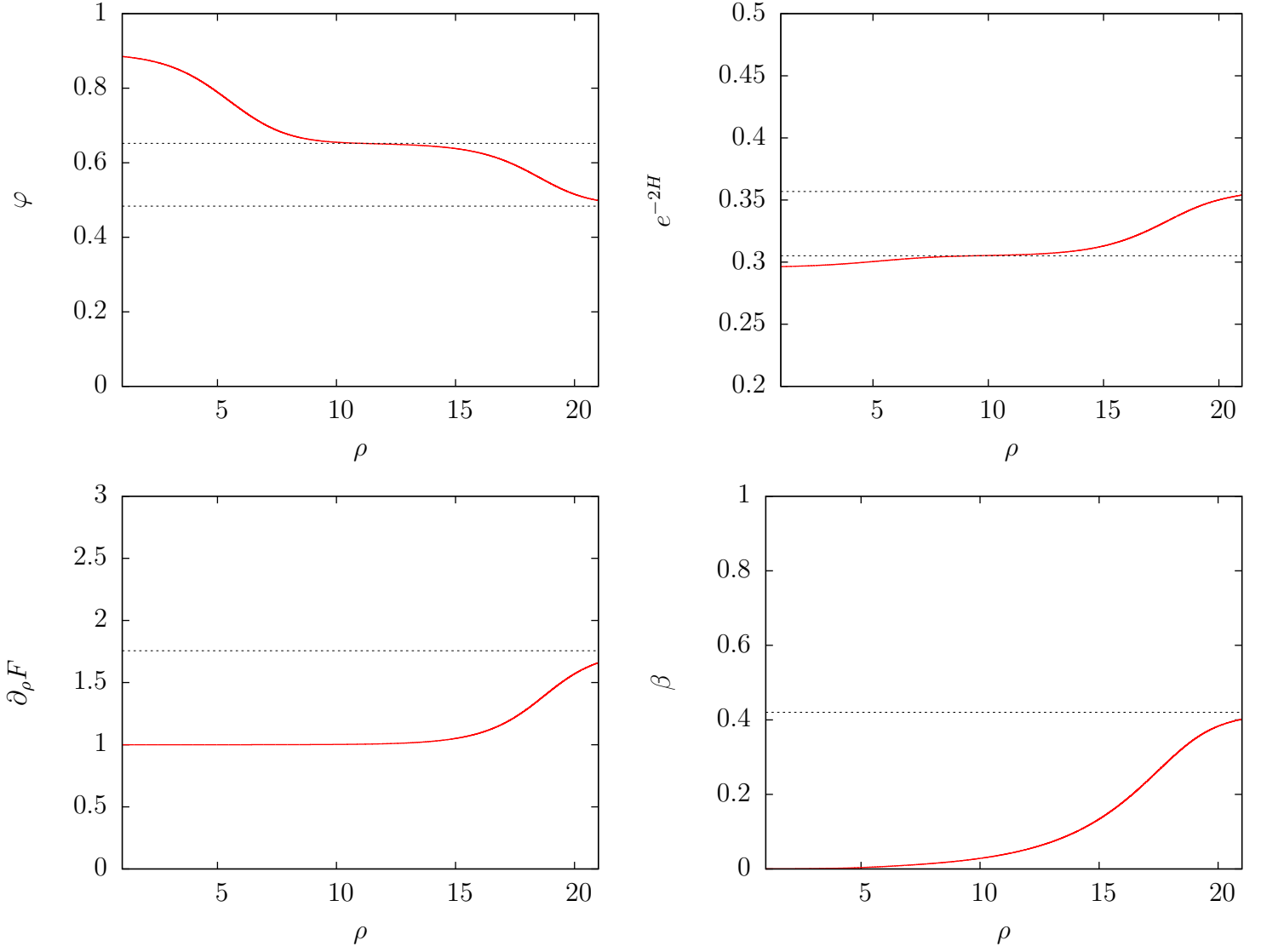


Figure 17: Holographic RG flow from an small z Lifshitz space with $z = 1.756$ to a point close to the AdS space on the small φ_0 branch with $\varphi_0 = 0.234$, and finally to the AdS space on the large φ_0 branch with $\varphi_0 = 0.8$. The dashed lines show the exact field values of the Lifshitz space.

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